CENTRAL BAGS AND RELATED TOPICS

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Contents

1. Introduction

This is a lightly edited excerpt from my PhD thesis that describes the *central bag method*, what I spent most of my PhD working on. The central bag method is a way to reduce a problem about a graph G to a problem about a proper induced subgraph β of G. The idea behind the central bag method is to make tractable a difficult problem by considering, instead of G, a graph that has stronger properties than G. The set-up of the method allows us to arrange for β to satisfy certain additional conditions; this lets us exploit the structure of G and β to guarantee that the problem in question is easier to solve in β than in G. Under the right conditions, we can then lift the solution in β to a solution in G.

The central bag method appears in [7, 2, 5, 6, 1, 4, 3] and other papers, as do several of the results in this writeup. I am sharing this presentation of the central bag method because it aims to generalize the definitions and tools used in the previously-cited papers in a unified and consistent way. We begin first by reviewing the essential background, then by presenting an overview of the intuition behind the central bag method.

2. Background

A *tree decomposition* (T, χ) of a graph G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$, satisfying the following properties:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$,
(ii) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$,
(iii) For every $v \in V(G)$, the support of v in (T, χ) is connected.

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The *width* of a tree decomposition (T, χ) is $\min_{t \in V(T)} |\chi(t)| - 1$. The *treewidth* of a graph G is the minimum width of a tree decomposition of G. The sets $\chi(t)$ for $t \in V(T)$ are called the *bags* of the tree decomposition (T, χ). For a set $X \subseteq V(T)$, we let $\chi(X) = \bigcup_{t \in X} \chi(t)$. An *adhesion* of a tree decomposition (T, χ) is a set of the form $\chi(t) \cap \chi(t')$ for $tt' \in E(T)$. For a vertex $t \in V(T)$, the *adhesions* of $\chi(t)$ are the sets $\chi(t) \cap \chi(t')$ for all $t' \in N(t)$. For a vertex $t \in V(T)$, the *torso of $\chi(t)$* is the graph $G(\chi(t))$ such that $V(G(\chi(t))) = \chi(t)$ and $E(G(\chi(t))) = E(G[\chi(t)]) \cup \left(\bigcup_{t' \in N(t)} \binom{\chi(t) \cap \chi(t')}{2} \right)$. We say that (T, χ) has *adhesion k* if $|\chi(t) \cap \chi(t')| \leq k$ for all $tt' \in E(T)$.

Let G be a graph. A function $w : V(G) \to \mathbb{R}$ is called a *weight function on G*. If w is a weight function on G, then for all $X \subseteq V(G)$, we say $w(X) := \sum_{x \in X} w(x)$. A weight function w on G is a *normal weight function* if $w : V(G) \to [0, 1]$ and if $w(G) = 1$.

Let G be a graph and let w be a normal weight function on G. A set $X \subseteq V(G)$ is called a *w-balanced separator of G* if for every component D of $G \setminus X$, it holds that $w(D) \leq \frac{1}{2}$. Essentially, w-balanced separators split G into small components with respect to the weight function w. It turns out that the property of having w-balanced separators of small size for every normal weight function w is equivalent to the property of having bounded treewidth. Before we prove this equivalence, we give several important definitions.

Let G be a graph, let w be a normal weight function on G, and let (T, χ) be a tree decomposition of G. We say that (T, χ) is *w-unbalanced* if for every separation $(A, C, B) \in \tau((T, \chi))$ it holds that $w(A) > 1/2$ or $w(B) > 1/2$. Conversely, (T, χ) is *w-balanced* if there exists a separation $(A, C, B) \in \tau((T, \chi))$ such that $w(A) \leq 1/2$ and $w(B) \leq 1/2$. If (T, χ) is w-unbalanced, we define the tree $\overrightarrow{T}$ as the directed tree formed from T by directing each edge of T toward the side of its corresponding separation that has large weight. We call $\overrightarrow{T}$ the *w-direction of T*.

A *sink* of a directed graph is a vertex v such that every edge incident with v is directed towards v.

Proposition 2.1. *Let T be a directed tree. Then, T has a sink.*

Proof. Suppose for a contradiction that T does not have a sink. Let $P = p_1\text{-}\dots\text{-}p_k$ be a longest directed path in T. Since T does not have a sink, the vertex p_k has an out-neighbor u. Since P is a longest directed path of T, it follows that $u \in V(P)$. But now T contains a cycle, contradicting that T is a tree. □

Lemma 2.2. *Let G be a graph and let (T, χ) be a tree decomposition of G. Suppose w is a normal weight function on G such that (T, χ) is w-unbalanced. Let $\overrightarrow{T}$ be the w-direction of T. Then, there exists a vertex $r \in V(\overrightarrow{T})$ such that for every path P of T from $v \in V(\overrightarrow{T}) \setminus \{r\}$ to r, P is a directed path from v to r in $\overrightarrow{T}$.*

Proof. We begin by proving the following property of $\overrightarrow{T}$:

(1) *For each $v \in V(\overrightarrow{T})$, there is at most one vertex $u \in N(v)$ such that the edge vu is directed from v to u.*

Suppose for a contradiction that there exist $u_1, u_2 \in N(v)$ such that the edge vu_1 is directed toward u_1 and the edge vu_2 is directed toward u_2. Let $S_1 = S_{vu_1}$ be the separation of G corresponding to edge vu_1 and let $S_2 = S_{vu_2}$ be the separation of G corresponding to edge vu_2. Let T_1 and T_2 be the components of $\overrightarrow{T} \setminus \{v\}$ containing u_1 and u_2, respectively. Because edge vu_1 is directed from v to u_1, it follows that $w(\chi(T_1) \setminus (\chi(u_1) \cap \chi(v))) > 1/2$. Similarly, $w(\chi(T_2) \setminus \chi(u_2) \cap \chi(v))) > 1/2$. However, by condition (iii) of the definition of tree decomposition, it follows that $\chi(T_1) \cap \chi(T_2) \subseteq \chi(u_1) \cap \chi(v) \cap \chi(u_2)$, and so $\chi(T_1) \setminus (\chi(u_1) \cap \chi(v))$ is disjoint from $\chi(T_2) \setminus (\chi(u_2) \cap \chi(v))$. But now $w(G) > 1$, contradicting that w is a normal weight function on G. This proves (1).

By Proposition 2.1, $\overrightarrow{T}$ has a sink $s \in V(T)$. We prove by induction on the length of the path that every path from a vertex $v \in V(\overrightarrow{T}) \setminus \{s\}$ to s is directed toward s. The base case is for paths of length one, which are the edges incident with s; by the definition of a sink, each of these paths is directed toward s. Assume every path of length $k-1$ from a vertex of $V(\overrightarrow{T}) \setminus \{s\}$ to s is directed toward s, and consider a path $P = v\text{-}p_1\text{-}\dots\text{-}p_{k-2}\text{-}s$ from $v \in V(\overrightarrow{T})$ to s of length k. Let p_1 be the neighbor of v in P. By the inductive hypothesis, the path $p_1\text{-}\dots\text{-}p_{k-2}\text{-}s$ is directed toward s. By (1), since p_2 is an out-neighbor of p_1, it follows that the edge vp_1 is directed toward p_1. Therefore, P is directed toward s. This completes the proof. □

We call the vertex r as in Lemma 2.2 the *w-heavy vertex of T*, and the bag $\chi(r)$ the *w-heavy bag of (T, χ)*. We can now prove the following lemma (which first appeared in [9] as 2.5).

Lemma 2.3. *Let G be a graph and let k be a positive real number. Suppose that $\operatorname{tw}(G) \leq k$. Then, for every normal weight function w on G, there exists $X_w \subseteq V(G)$ such that X_w is a w-balanced separator of G of size at most $k+1$.*

Proof. Let (T, χ) be a tree decomposition of G of width k. First, suppose that (T, χ) is not w-unbalanced. Then, there is an edge $e = t_1t_2$ of T and separation $S_e = (A_e, C_e, B_e)$ such that $w(A_e) < \frac{1}{2}$ and $w(B_e) < \frac{1}{2}$. Also, $C_e \subseteq \chi(t_1) \cap \chi(t_2)$, so $|C_e| \leq k+1$. Now, $X_w = C_e$ is a w-balanced separator of G of size at most $k+1$.

Therefore, we may assume that (T, χ) is w-unbalanced. Let $r \in V(T)$ be the w-heavy vertex of T as in Lemma 2.2, and let $t_1, \dots, t_m$ be the neighbors of r in T. For each $i \in \{1, \dots, m\}$, let $S_i = S_{rt_i} = (A_i, C_i, B_i)$, and assume up to symmetry between A_i and B_i that $w(A_i) < 1/2$. Now, the connected components of $G \setminus \chi(r)$ are exactly the sets $A_1, \dots, A_m$. It follows that $X_w = \chi(r)$ is a w-balanced separator of G of size at most $k+1$. This completes the proof. □

The converse of Lemma 2.3 also holds:

Lemma 2.4 ([8]; [2], Lemma 1.13)**.** *Let G be a graph and let k be a positive real number. Suppose that G has a w-balanced separator of size at most k for every normal weight function w on G. Then,* $\text{tw}(G) \leq 2k$.

Because of Lemmas 2.3 and 2.4, we can use the existence of w-balanced separators of bounded size as a characterization of bounded treewidth. This characterization is useful because separators are generally better-understood and more intuitive than tree decompositions, and thus easier to work with. In this thesis, the relationship between balanced separators and treewidth is crucial.

Balanced separators are not the only graph parameter that interacts well with treewidth. Indeed, treewidth is related to many important graph structures and parameters. The paper [8] is an extensive survey of graph parameters that are tied to treewidth.

Let G be a graph. A *separation* of G is a triple (A, C, B) such that A, C, and B are disjoint, $A \cup C \cup B = V(G)$, and A is anticomplete to B. If G is connected, (A, C, B) is a separation of G, and A and B are both non-empty, then C is a cutset of G. For a separation $S = (A, C, B)$, we set the notation $A(S) \coloneqq A$, $C(S) \coloneqq C$, and $B(S) \coloneqq B$. First, we show the following:

Proposition 2.5. *Let G be a graph and let (T, χ) be a tree decomposition of G. Let $e = t_1t_2$ be an edge of $E(T)$ and let T_1, T_2 be the connected components of $T \setminus \{e\}$ containing t_1 and t_2, respectively. Then, $\chi(T_1) \setminus (\chi(t_1) \cap \chi(t_2))$ and $\chi(T_2) \setminus (\chi(t_1) \cap \chi(t_2))$ are disjoint and anticomplete to each other.*

Proof. Let $X_1 = \chi(T_1) \setminus (\chi(t_1) \cap \chi(t_2))$ and $X_2 = \chi(T_2) \setminus (\chi(t_1) \cap \chi(t_2))$. First, we show that X_1 and X_2 are disjoint. Suppose that $x \in X_1 \cap X_2$. Then, $x \in \chi(T_1)$ and $x \in \chi(T_2)$. By condition (iii) of the definition of tree decomposition, it follows that $x \in \chi(t_1)$ and $x \in \chi(t_2)$, a contradiction. Therefore, X_1 and X_2 are disjoint.

Next, suppose $x_1 \in X_1$, $x_2 \in X_2$, and x_1x_2 is an edge of G. By condition (ii) of the definition of tree decomposition, it follows that there exists $t \in V(T)$ such that $\{x_1, x_2\} \subseteq \chi(t)$. Assume up to symmetry between T_1 and T_2 that $t \in T_1$. Then, $x_2 \in \chi(T_1)$, so by condition (iii) of the definition of tree decomposition, $x_2 \in \chi(t_1)$ and $x_2 \in \chi(t_2)$, a contradiction. □

Let G be a graph and let (T, χ) be a tree decomposition of G. Let $e = t_1t_2$ and let $C_e = \chi(t_1) \cap \chi(t_2)$, $A_e = \chi(T_1) \setminus C_e$, and $B_e = \chi(T_2) \setminus C_e$. Now, A_e and B_e are anticomplete to each other by Proposition 2.5. The *separation corresponding to e*, denoted S_e, is defined as $S_e = (A_e, C_e, B_e)$ (with symmetry between t_1 and t_2 and thus A_e and B_e). The correspondence between an edge of the tree decomposition and a separation is visualized in Figure 1. The *collection of separations corresponding to (T, χ)*, denoted $\tau((T, \chi))$ is defined as follows: $\tau((T, \chi)) = \{S_e \mid e \in E(T)\}$.

Two separations $S_1 = (A_1, C_1, B_1)$ and $S_2 = (A_2, C_2, B_2)$ are called *non-crossing* if up to symmetry between A and B it holds that $A_1 \cup C_1 \subseteq B_2 \cup C_2$ and $A_2 \cup C_2 \subseteq B_1 \cup C_1$. A collection $\mathcal{S}$ of separations of G is called *laminar* if the separations of $\mathcal{S}$ are pairwise

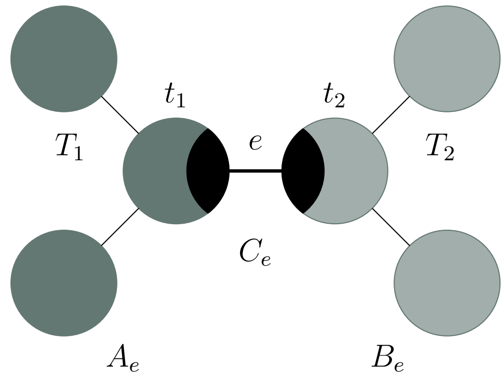

FIGURE 1. A visualization of the correspondence between the edge $e = t_1t_2$ (shown in bold) and the separation (A_e, C_e, B_e), where A_e is mint, C_e is black, and B_e is light mint.

non-crossing. The following crucial result of Robertson and Seymour states that there is a correspondence between tree decompositions of G and laminar collections of separations of G.

Theorem 2.6 ([10], 9.1)**.** *Let G be a graph. Then, for every tree decomposition (T, χ) of G, $\tau((T, \chi))$ is laminar. Conversely, for every laminar collection of separations $\mathcal{S}$ of G, there exists a tree decomposition (T, χ) of G such that $\tau((T, \chi)) = \mathcal{S}$.*

3. Overview

Let G be a graph, and assume that our task is to determine the treewidth of G (up to a constant factor; mainly, we are interested in whether or not a bound exists, not in optimizing the bound). Recall that a tree decomposition (T, χ) of G has *adhesion k* if $|\chi(u) \cap \chi(v)| \leq k$ for all $uv \in E(T)$. Suppose that we are given a tree decomposition (T, χ) of G of adhesion k. There are two possibilities:

(1) The tree decomposition (T, χ) is w-balanced for every normal weight function w on G; or
(2) there exists a normal weight function w on G such that (T, χ) is w-unbalanced.

Suppose (1) holds. Then, for every normal weight function w on G, there is a separation $(A_w, C_w, B_w) \in \tau((T, \chi))$ of G such that $w(A_w) \leq 1/2$, $w(B_w) \leq \frac{1}{2}$, and $C_w \subseteq \chi(t) \cap \chi(t')$ for some edge $tt' \in E(T)$. Since (T, χ) has adhesion k, it follows that $|C_w| \leq k$, so G has a w-balanced separator of size at most k for every normal weight function w on G. By Lemma 2.4, we conclude that $\text{tw}(G) \leq 2k$. Therefore, we may assume that (2) holds.

Let w be a normal weight function on G such that (T, χ) is w-unbalanced. Then, we can form the w-direction $\overrightarrow{T}$ of T, and by Lemma 2.2, there is a w-heavy vertex $r \in V(T)$.

Let $\chi(r)$ be the w-heavy bag of T, and, following Figure 2, let T_1, T_2, T_3 be the components of $T \setminus \{r\}$ rooted at t_1, t_2, t_3, respectively. Now, $G \setminus \chi(r)$ consists of connected components D such that D has small weight and small neighborhood in β; specifically, $w(D) < \frac{1}{2}$ and $|N(D)| \leq k$. These connected components are obtained by considering the separations given by (T, χ) from the edges t_1r, t_2r, and t_3r. The crucial observation for the central bag

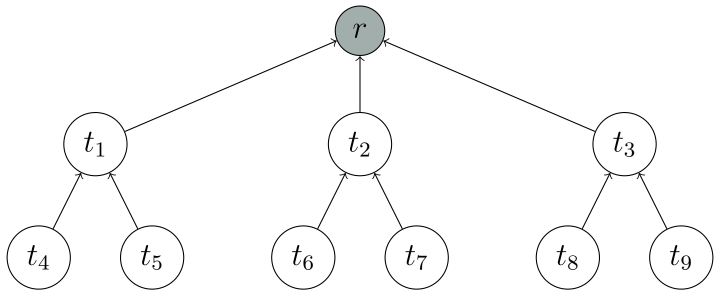

FIGURE 2. An example of the tree $\overrightarrow{T}$ with heavy vertex r.

method is that since the components of $G \setminus \chi(r)$ are all of small weight and attach to $\chi(r)$ in a controlled way, they don't contribute very much to the treewidth of G. Given some additional assumptions, we can show that the treewidth of G is bounded by a function of the treewidth of $\chi(r)$.

Now, the task of finding the treewidth of G is reduced to the task of finding the treewidth of $\chi(r)$. Therefore, we want $\chi(r)$ to have some additional properties relative to G that make determining the treewidth an easier problem for $\chi(r)$ than for G. How can we guarantee that $\chi(r)$ has these additional properties? In this overview, we started from a tree decomposition of G. Recall that tree decompositions are equivalent to laminar collections of separations. In practice, we usually start not with a tree decomposition but instead with a collection of separations that was chosen with respect to the graph G. Most graphs that we are interested in have *structure theorems* that describe the existence and properties of cutsets in the graph. The idea is to use these cutsets to define laminar collections of separations that allow us to draw strong conclusions about the structure of $\chi(r)$.

When we state and prove the central bag method formally in the next section, everything will be in the language of separations rather than tree decompositions. This is mostly because we can relax some properties of separations and maintain the strength of the central bag method, but the relaxations cost us the direct correspondence with tree decompositions. Separations are also more natural graph structures to work with, and interact well with existing structure theorems. However, I find that the basic intuition of reducing to a bag of a well-chosen tree decomposition is valuable and helps to illustrate several of the properties we care about while using the central bag method. Therefore, we will return to this informal overview at the end of the next section to help motivate certain definitions and properties.

4. The details

Let G be a graph and let $w : V(G) \to [0, 1]$ be a normal weight function on G. Let $\mathcal{S}$ be a collection of separations of G. We define the *central bag for $\mathcal{S}$*, denoted $\beta_{\mathcal{S}}$, as

$$\beta_{\mathcal{S}} = \bigcap_{S \in \mathcal{S}} (B(S) \cup C(S)). \tag{1}$$

Observe that $G \setminus \beta_{\mathcal{S}} = \bigcup_{S \in \mathcal{S}} A(S)$.

A collection $\mathcal{S}$ of separations of G is *k-aligned* (with respect to a normal weight function w on G) if the following conditions hold:

(1) For all $S \in \mathcal{S}$:
  (i) $C(S) \cap \beta_{\mathcal{S}}$ is connected;
  (ii) there is a set $\delta(S) \subseteq C(S) \cap \beta_{\mathcal{S}}$ such that $|\delta(S)| \leq k$ and $(A(S) \cup (C(S) \setminus \delta(S)), \delta(S), B(S))$ is a separation of G;
  (iii) $w(A(S) \cup (C(S) \setminus \delta(S))) < \frac{1}{2}$; and
(2) for every component D of $\bigcup_{S \in \mathcal{S}} A(S)$, there exists $S \in \mathcal{S}$ such that $D \subseteq A(S)$.

The definition of k-aligned makes several implicit assumptions about collections of separations. Suppose $\mathcal{S}$ is k-aligned and $S \in \mathcal{S}$. Because of condition (1iii) of the definition of k-aligned, it holds that $w(A) < 1/2$. In applications of the central bag method, the way we typically guarantee that condition (1iii) is satisfied is by arranging that $w(B) > 1/2$ for the separations (A, C, B) that we work with. In general, in the remainder of this thesis, we assume by convention for all separations (A, C, B) that $w(A) \leq w(B)$, and in most cases that $w(A) < 1/2$ and $w(B) > 1/2$.

By fixing that $w(A) \leq w(B)$, the symmetry between A and B in the separation (A, C, B) is broken. We can thus update the definition of laminar accordingly. Two separations (A_1, C_1, B_1) and (A_2, C_2, B_2) are *A-non-crossing* if $A_1 \cap A_2 = C_1 \cap A_2 = A_1 \cap C_2 = \emptyset$. Figure 3 gives an illustration of the definition of A-non-crossing. A collection $\mathcal{S}$ of separations is *A-laminar* if the separations in $\mathcal{S}$ are pairwise A-non-crossing.

	A_1	C_1	B_1
A_2	$\emptyset$	$\emptyset$	
C_2	$\emptyset$		
B_2			

FIGURE 3. Two separations $S_1 = (A_1, C_1, B_2)$ and $S_2 = (A_2, C_2, B_2)$ are A-non-crossing if $A_1 \cap A_2 = A_1 \cap C_2 = A_2 \cap C_1 = \emptyset$.

Condition (2) of the definition of k-aligned is a relaxation of being A-laminar. Indeed, if a collection $\mathcal{S}$ is A-laminar, then it satisfies condition (2) of the definition of k-aligned:

Lemma 4.1. *Let G be a graph and let $\mathcal{S}$ be an A-laminar collection of separations. Then, $\mathcal{S}$ satisfies condition (2) of the definition of k-aligned.*

Proof. Let D be a component of $\bigcup_{S \in \mathcal{S}} A(S)$ and let $S_1 \in \mathcal{S}$ be such that $D \cap A(S_1) \neq \emptyset$. Now, $N(A(S_1)) \subseteq C(S_1)$ and, because $\mathcal{S}$ is A-laminar, $C(S_1) \cap A(S_2) = \emptyset$ for all $S_2 \in \mathcal{S}$. Therefore, $D \subseteq A(S_1)$. This completes the proof. □

When $\mathcal{S}$ is k-aligned, we define, for all $S \in \mathcal{S}$, $A^*(S) \coloneqq A(S) \cup (C(S) \setminus \delta(S))$. We also define an *anchor map* $\delta^* : \mathcal{S} \to V(G)$ with $\delta^*(S) \in \delta(S)$ for all $S \in \mathcal{S}$. For all $S \in \mathcal{S}$, we call $\delta^*(S)$ the *anchor for S*.

Lemma 4.2. *Let G be a graph and let w be a normal weight function on G. Let $\mathcal{S}$ be a k-aligned collection of separations of G. Then, the anchor for each separation of $\mathcal{S}$ is in $\beta_{\mathcal{S}}$. Also, the function $\delta^{*-1} : \delta^*(\mathcal{S}) \to 2^{\mathcal{S}}$ is well-defined (where $\delta^{*-1}(v) = \{S \in \mathcal{S} \mid \delta^*(S) = v\}$) and $\{\delta^{*-1}(v) \mid v \in V(G)\}$ is a partition of $\mathcal{S}$.*

Proof. Let $S \in \mathcal{S}$. By (1ii) of the definition of k-aligned, it follows that $\delta(S) \subseteq \beta_{\mathcal{S}}$, and by construction $\delta^*(S) \in \delta(S)$, so the anchor for S is in $\beta_{\mathcal{S}}$. The function δ^{*-1} is well-defined where $\delta^{*-1}(v)$ is the set of separations $S \in \mathcal{S}$ such that $\delta^*(S) = v$. Since δ^* is a function, it follows that for every $S \in \mathcal{S}$, there is exactly one v such that $S \in \delta^{*-1}(v)$. Thus, $\{\delta^{*-1}(v) \mid v \in V(G)\}$ is a partition of $\mathcal{S}$. This completes the proof. $\square$

We also define a weight function $w_{\mathcal{S}}$ on $\beta_{\mathcal{S}}$ called the *inherited weight function for $\mathcal{S}$*. Let $\mathcal{O}$ be a fixed ordering of the vertices $V(G)$. For each component D of $\bigcup_{S \in \mathcal{S}} A(S)$, let $f(D)$ denote the $\mathcal{O}$-minimum vertex v such that $D \subseteq \bigcup_{S \in \delta^{*-1}(v)} A(S)$. Let $A_{\mathcal{O}}(v)$ denote the union of the components D of $\bigcup_{S \in \mathcal{S}} A(S)$ such that $f(D) = v$. By Lemma 4.2, the anchor for every separation in $\mathcal{S}$ is in $\beta_{\mathcal{S}}$, and so $\{A_{\mathcal{O}}(v) \mid v \in \beta_{\mathcal{S}}\}$ is a partition of $\bigcup_{S \in \mathcal{S}} A(S)$. (Possibly $A_{\mathcal{O}}(v) = \{\}$). Now, for all $v \in \beta_{\mathcal{S}}$,

$$w_{\mathcal{S}}(v) = w(v) + w(A_{\mathcal{O}}(v)). \tag{2}$$

The inherited weight function $w_{\mathcal{S}}$ depends on the anchor map δ^* for $\mathcal{S}$; the weight function is essentially "inherited from" the anchor map. Next, we show that $w_{\mathcal{S}}$ is a normal weight function on $\beta_{\mathcal{S}}$.

Lemma 4.3. *Let G be a graph and let w be a normal weight function on G. Let $\mathcal{S}$ be a k-aligned collection of separations of G. Then, the inherited weight function $w_{\mathcal{S}}$ for $\mathcal{S}$ is a normal weight function on the central bag $\beta_{\mathcal{S}}$ for $\mathcal{S}$.*

Proof. By the definition of $w_{\mathcal{S}}$, and since w is a normal weight function, it follows that $w_{\mathcal{S}} : \beta_{\mathcal{S}} \to [0,1]$. We need to show that $w_{\mathcal{S}}(\beta_{\mathcal{S}}) = 1$. By the definition of $w_{\mathcal{S}}$, we have

$$\begin{aligned} w_{\mathcal{S}}(\beta_{\mathcal{S}}) &= \sum_{v \in \beta_{\mathcal{S}}} w_{\mathcal{S}}(v) \\ &= \sum_{v \in \beta_{\mathcal{S}}} w(v) + \sum_{v \in \beta_{\mathcal{S}}} w(A_{\mathcal{O}}(v)). \end{aligned}$$

By Lemma 4.2 and the definition of $A_{\mathcal{O}}$, it follows that $\bigcup_{v \in \beta_{\mathcal{S}}} A_{\mathcal{O}}(v) = \bigcup_{S \in \mathcal{S}} A(S)$. Therefore,

$$\begin{aligned} w_{\mathcal{S}}(\beta_{\mathcal{S}}) &= \sum_{v \in \beta_{\mathcal{S}}} w(v) + \sum_{S \in \mathcal{S}} w(A(S)) \\ &= \sum_{v \in \beta_{\mathcal{S}}} w(v) + \sum_{v \in V(G) \setminus \beta_{\mathcal{S}}} w(v) \\ &= \sum_{v \in V(G)} w(v) \\ &= 1, \end{aligned}$$

where the last equality holds since w is a normal weight function on G. This completes the proof. $\square$

Next, we prove that there exist conditions under which it holds that if $\beta_{\mathcal{S}}$ has a $w_{\mathcal{S}}$-balanced separator of bounded size, then G has a w-balanced separator of bounded size.

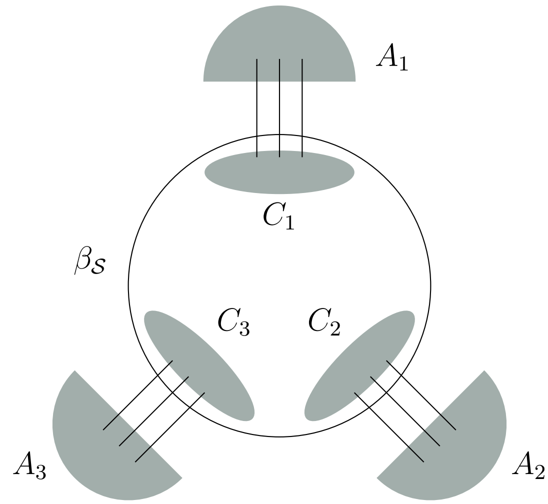

FIGURE 4. An illustration of the central bag $\beta_{\mathcal{S}}$ for a k-aligned collection $\mathcal{S}$ with three separations (A_1, C_1, B_1), (A_2, C_2, B_2), and (A_3, C_3, B_3).

We first need the following definition. Let $\mathcal{S}$ be a k-aligned collection of separations with anchor map δ^* and let $X \subseteq \beta_{\mathcal{S}}$. . A separation $S \in \mathcal{S}$ is said to *cross* X if either $\delta^*(S) \in X$ or if there exist two distinct components D_1, D_2 of $\beta_{\mathcal{S}} \setminus X$ such that $C(S) \cap D_1 \neq \emptyset$ and $C(S) \cap D_2 \neq \emptyset$.

Theorem 4.4. *Let G be a graph and let w be a normal weight function on G. Let $\mathcal{S}$ be a k-aligned collection of separations of G, let δ^* be the anchor map for $\mathcal{S}$, and let $w_{\mathcal{S}}$ be the inherited weight function for $\mathcal{S}$. Suppose that $\beta_{\mathcal{S}}$ has a balanced separator X of size γ. Let c denote the number of separations of $\mathcal{S}$ that cross X in $\beta_{\mathcal{S}}$. Then, G has a w-balanced separator of size $\gamma + ck$.*

Proof. Let $\mathcal{S}' \subseteq \mathcal{S}$ denote the set of all separations S of $\mathcal{S}$ such that $C(S)$ crosses X. Let Y be defined as follows:

$$Y = X \cup \left(\bigcup_{S \in \mathcal{S}'} \delta(S) \right). \tag{3}$$

By condition (1ii), $\delta(S) \subseteq \beta_{\mathcal{S}}$ for all $S \in \mathcal{S}$, so $Y \subseteq \beta_{\mathcal{S}}$. Next, we claim that Y is a w-balanced separator of G of size at most $\gamma + ck$. First, we show that Y has size at most $\gamma + ck$.

(2) $|Y| \leq \gamma + ck$.

By the definition, $|Y| \leq |X| + |\mathcal{S}'| \cdot \max_{S \in \mathcal{S}'} |\delta(S)|$. Since $\mathcal{S}$ is k-aligned, it follows that $|\delta(S)| \leq k$ for all $S \in \mathcal{S}'$. By the assumptions of the theorem, $|X| \leq \gamma$ and $|\mathcal{S}'| \leq c$. Therefore, $|Y| \leq \gamma + ck$. This proves (2).

Next, we show that Y is a w-balanced separator of G.

(3) *Let M be a connected component of $G \setminus Y$. Then, $w(M) \leq \frac{1}{2}$.*

Suppose $w(M) > \frac{1}{2}$. Let $Q_1, \dots, Q_m$ be the connected components of $\beta_{\mathcal{S}} \setminus X$ and let $D_1, \dots, D_\ell$ be the connected components of $G \setminus \beta_{\mathcal{S}}$. Note that $M \subseteq (\bigcup_{i=1}^{m} Q_i) \cup \left(\bigcup_{i=1}^{\ell} D_i\right)$ and that, since $Y \subseteq \beta_{\mathcal{S}}$, if $D_i \cap M \neq \emptyset$, then $D_i \subseteq M$. By condition (2) of the definition of k-aligned, it follows that for every $1 \leq i \leq \ell$ there exists $S_i \in \mathcal{S}$ such that $D_i \subseteq A(S_i)$.

First, suppose that $Q_1 \cap M \neq \emptyset$ and $Q_2 \cap M \neq \emptyset$. Since Q_1 and Q_2 are connected components of $\beta_{\mathcal{S}} \setminus X$, it follows that there exists D_i such that $D_i \cap M \neq \emptyset$, $C(S_i) \cap Q_1 \neq \emptyset$, and $C(S_i) \cap Q_2 \neq \emptyset$. In particular, $C(S_i)$ crosses X, and so $\delta(S_i) \subseteq Y$.

By condition (1ii) of the definition of k-aligned, it holds that $(A^*(S_i), \delta(S_i), B(S_i))$ is a separation of G. Since $M \cap A^*(S_i) \neq \emptyset$ and $\delta(S_i) \subseteq Y$, it follows that $M \subseteq A^*(S_i)$. But by condition (1iii) of the definition of k-aligned, we conclude that $w(M) \leq w(A^*(S_i)) \leq \frac{1}{2}$, a contradiction.

Therefore, we may assume that $M \cap Q_i = \emptyset$ for $2 \leq i \leq m$. Suppose that $Q_1 \cap M = \emptyset$. It follows that there exists $1 \leq i \leq \ell$ such that $M = D_i$. But since $D_i \subseteq A(S_i)$ and $w(A(S_i)) \leq \frac{1}{2}$, it follows that $w(M) \leq \frac{1}{2}$, a contradiction. Therefore, $Q_1 \cap M \neq \emptyset$. Let

$$M' = Q_1 \cup \bigcup_{S \in \mathcal{S}, C(S) \subseteq Q_1 \cup X} A^*(S).$$

By definition, $M \subseteq M'$. Also,

$$\begin{aligned} w(M) &\leq w(M') \\ &= w(Q_1) + \sum_{S \in \mathcal{S}, C(S) \subseteq Q_1 \cup X} w(A^*(S)) \\ &\leq w_{\mathcal{S}}(Q_1) \\ &\leq \frac{1}{2}, \end{aligned}$$

a contradiction. This proves (3).

We have now shown that Y is a w-balanced separator of G of size at most $\gamma + ck$. This completes the proof. □

Theorem 4.4 is the reason that the central bag method is a powerful tool to study treewidth: given a collection of k-aligned separations $\mathcal{S}$ and an anchor map δ^* for $\mathcal{S}$, Theorem 4.4 allows us to bound the treewidth of a graph G by a function of the treewidth of the central bag $\beta_{\mathcal{S}}$. (We move between balanced separators and treewidth using Lemmas 2.3 and 2.4). The bound on the treewidth of G depends on three key pieces:

(1) the treewidth of $\beta_{\mathcal{S}}$,
(2) the smallest k such that $\mathcal{S}$ is k-aligned, and
(3) the number of separations in $\mathcal{S}$ that cross a balanced separator of $\beta_{\mathcal{S}}$.

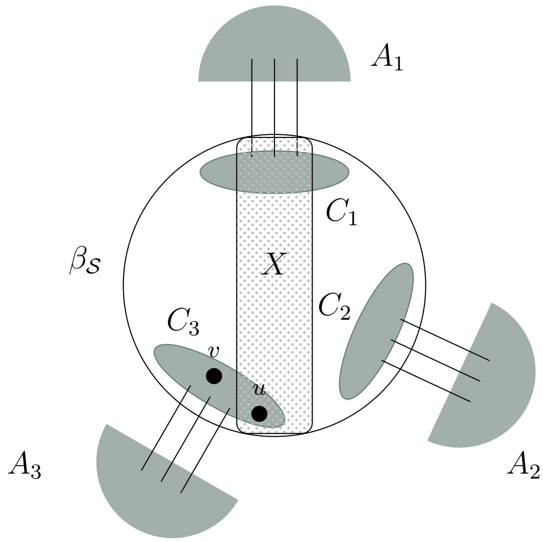

FIGURE 5. An illustration of a balanced separator X in a central bag. Suppose $\beta_{\mathcal{S}} \setminus X$ has two connected components, the left and right halves of the circle. Then, the separation (A_1, C_1, B_1) crosses X, but (A_2, C_2, B_2) does not. The separation $S_3 = (A_3, C_3, B_3)$ crosses X if $\delta(S_3) = u$, but not if $\delta(S_3) = v$.

The task of applying the central bag method to prove bounded-treewidth results therefore amounts to arranging these three properties favorably according to the problem at hand.

Let's now revisit the informal overview involving the heavy bag of a tree decomposition from the beginning of the chapter. In particular, let's prove that the set-up from before fits into the framework defined in this section.

Theorem 4.5. *Let G be a graph and let (T, χ) be a tree decomposition of G of adhesion k. Assume that w is a normal weight function on G such that (T, χ) is w-unbalanced. Let $\mathcal{S} = \tau((T, \chi))$. Let r be the w-heavy vertex of T as in Lemma 2.2, and let $\chi(r)$ be the w-heavy bag of (T, χ). Then, there exists $\mathcal{S}' \subseteq \mathcal{S}$ such that $\chi(r) = \beta_{\mathcal{S}'}$ and $\mathcal{S}'$ satisfies conditions (1ii), (1iii), and (2) of the definition of k-aligned.*

Proof. Let $t_1, \dots, t_m$ be the neighbors of r in T, and let $\mathcal{S}' \subseteq \mathcal{S}$ be the collection of separations corresponding to the edges $t_1r, \dots, t_mr$. First, we show that $\chi(r) = \beta_{\mathcal{S}'}$. Since (T, χ) is w-unbalanced, up to symmetry between A and B we may assume (following our usual convention) that $w(A(S)) < 1/2$ and $w(B(S)) > 1/2$ for every separation $S \in \mathcal{S}'$. Let $\mathcal{S}' = S_1, \dots, S_m$, where S_i corresponds to the edge t_ir. Let T_i be the component of $T \setminus \{t_ir\}$ containing r for $i \in \{1, \dots, m\}$. By the definition of w-heavy bag, $\chi(T_i) = B(S_i) \cup C(S_i)$ for $i \in \{1, \dots, m\}$. Now, $\chi(r) = \bigcap_{i=1}^{m} \chi(T_i) = \bigcap_{S \in \mathcal{S}'} (B(S) \cup C(S)) = \beta_{\mathcal{S}'}$.

Next, we consider the conditions of the definition of k-aligned. Since (T, χ) has adhesion k, it follows that $|C(S)| \leq k$ for all $S \in \mathcal{S}$, so condition (1ii) holds with $\delta(S) = C(S)$ for all $S \in \mathcal{S}'$ (note that for every $S \in \mathcal{S}$, $C(S) \subseteq \chi(r) = \beta_{\mathcal{S}'}$). Condition (1iii) holds since $w(A(S)) < 1/2$ for all $S \in \mathcal{S}$. By Theorem 2.6, $\mathcal{S}$ is laminar, so $\mathcal{S}'$ is laminar. Since $\chi(r) = \bigcap_{S \in \mathcal{S}'} (B(S) \cup C(S))$, it follows that $\mathcal{S}'$ is A-laminar. Now, condition (2) holds by Lemma 4.1. This completes the proof. $\square$

The proof of Theorem 4.4 illustrates the motivation behind the conditions included in the definition of k-aligned, and Theorem 4.5 reveals the inspiration behind the definition of central bag. The only condition not mentioned in either proof is condition (1i), and indeed, all of the machinery developed so far works perfectly well even if condition (1i) is excluded. However, condition (1i) is necessary in every application to help bound the number of separations of $\mathcal{S}$ that cross a balanced separator of $\beta_{\mathcal{S}}$. Since condition (1i) is always required for this purpose (at least in currently-known applications), we include it as part of the definition of k-aligned here.

The proof of Theorem 4.5 also illustrates another important concept. Recall that $\mathcal{S}'$ consisted of only the separations corresponding to edges of T incident with the w-heavy vertex r. This is equivalent to the following observation. Let $\mathcal{S}$ be a collection of separations and let $S, S' \in \mathcal{S}$ such that $B(S) \cup C(S) \subseteq B(S') \cup C(S')$. Then, $\beta_{\mathcal{S}} = \beta_{\mathcal{S} \setminus \{S'\}}$. In other words, the separation S' is not an essential element of the collection $\mathcal{S}$. Under these conditions, we say that S is a *shield for S'*. We say that a collection of separations $\mathcal{S}$ is *shield-minimal* if $B(S) \cup C(S) \not\subseteq B(S') \cup C(S')$ for all distinct pairs $S, S' \in \mathcal{S}$. In the remainder of this thesis, we will normally restrict our attention to shield-minimal collections of separations.

5. Star cutsets

In this section, we explain how to use star cutsets to construct a collection of separations that can be used with the central bag method. A *star cutset* of a connected graph G is a set $C \subseteq V(G)$ such that $G \setminus C$ is not connected and there exists $x \in C$ such that $C \subseteq N[x]$. Star cutsets appear often in structure theorems for major graph classes. For example, they are a type of decomposition for both perfect graphs and even-hole-free graphs, two major hereditary graph classes that are considered in several results in this thesis. See [11] and [12] for surveys on perfect graphs and even-hole-free graphs, respectively; both surveys highlight the ways that star cutsets contribute to the respective structure theorems. In this section, we don't yet mention the connection between star cutsets and structure theorems, but instead focus on properties of star cutsets and how they interact with the central bag method.

Let G be a graph and let w be a normal weight function on G. A vertex $v \in V(G)$ is called *w-unbalanced* if there is a component D of $G \setminus N[v]$ such that $w(D) > \frac{1}{2}$. A vertex v is *w-balanced* if it is not w-unbalanced. Let v be a w-unbalanced vertex of G. The *canonical star separation for v*, denoted $S(v) = (A(v), C(v), B(v))$, is defined as follows: $B(v)$ is the component of $G \setminus N[v]$ with $w(B(v)) > \frac{1}{2}$, $C(v) = N(B(v)) \cup \{v\}$, and $A(v) = V(G) \setminus (B(v) \cup C(v))$.

Lemma 5.1. *Let G be a graph and let w be a normal weight function on G. Let U be the set of w-unbalanced vertices of G. Let $x, y \in U$ such that $x \in A(y)$. Then, $B(y) \subseteq B(x)$ and $A(x) \subseteq A(y) \cup \{y\}$.*

Proof. Since $x \in A(y)$ and $N(A(y)) \subseteq C(y)$, it follows that $N[x] \subseteq A(y) \cup C(y)$. Therefore, $B(y)$ is contained in a component of $G \setminus N[x]$, and since $w(B(y)) > \frac{1}{2}$, it follows that $B(y) \subseteq B(x)$. For every $u \in C(y) \setminus \{y\}$, it holds that u has a neighbor in $B(y)$ and thus in

$B(x)$. Since $A(x)$ is anticomplete to $B(x)$, it follows that $(C(y)\setminus\{y\})\cap A(x)=\emptyset$. Therefore, $A(x)\subseteq A(y)\cup\{y\}$. □

Fix an ordering $\mathcal{O}$ of $V(G)$. Let $U\subseteq V(G)$ be the set of w-unbalanced vertices of G. Two vertices $x,y\in U$ are *star twins* if $B(x)=B(y)$, $C(x)\setminus\{x\}=C(y)\setminus\{y\}$, and $A(x)\cup\{x\}=A(y)\cup\{y\}$.

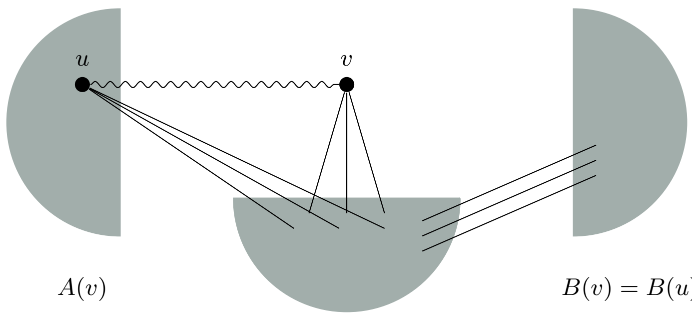

FIGURE 6. Star twins u and v, where u and v may or may not be adjacent

Lemma 5.2. *Let G be a graph and let w be a normal weight function on G. Let U be the set of w-unbalanced vertices of G. Let $x,y\in U$ such that $x\in A(y)$ and $y\in A(x)$. Then, x and y are star twins.*

Proof. Since $x\in A(y)$ and $A(y)$ is anticomplete to $B(y)$, it follows that $N[x]\subseteq A(y)\cup C(y)$. Therefore, $B(y)$ is contained in a connected component of $G\setminus N[x]$. Since $w(B(y))>\frac{1}{2}$, it follows that $B(y)\subseteq B(x)$. By symmetry, $B(x)\subseteq B(y)$, and so $B(x)=B(y)$.

Let $w\in C(y)\setminus\{y\}$. Since every vertex of $C(y)\setminus\{y\}$ has a neighbor in $B(y)$, and thus in $B(x)$, it follows that w has a neighbor in $B(x)$. Therefore, $w\in C(x)$, and so $C(y)\setminus\{y\}\subseteq C(x)$. By symmetry, $C(x)\setminus\{x\}\subseteq C(y)$. It follows that $C(x)\setminus\{x\}=C(y)\setminus\{y\}$. Since $A(x)\cup C(x)\cup B(x)=A(y)\cup C(y)\cup B(y)$, we conclude that $A(x)\cup\{x\}=A(y)\cup\{y\}$, and so x and y are star twins. □

Next, we define a relation $\leq^w_{\mathcal{O}}$ on U as follows: for $x,y\in U$,

$$x\leq^w_{\mathcal{O}} y \quad \text{if} \quad \begin{cases} x=y, \quad \text{or} \\ x \text{ and } y \text{ are star twins and } \mathcal{O}(x)<\mathcal{O}(y), \quad \text{or} \\ x \text{ and } y \text{ are not star twins and } y\in A(x). \end{cases}$$

Lemma 5.3. *Let G be a graph, let w be a normal weight function on G, and let $\mathcal{O}$ be an ordering of $V(G)$. Let $U\subseteq V(G)$ be the set of w-unbalanced vertices of G. Then, $\leq^w_{\mathcal{O}}$ is a partial order on U.*

Proof. We show that $\leq^w_{\mathcal{O}}$ is reflexive, antisymmetric, and transitive. By the definition, $\leq^w_{\mathcal{O}}$ is reflexive. Suppose $x\leq^w_{\mathcal{O}} y$ and $y\leq^w_{\mathcal{O}} x$ for $x,y\in U$. If x and y are star twins, then $x\leq^w_{\mathcal{O}} y$ implies that $\mathcal{O}(x)<\mathcal{O}(y)$, and $y\leq^w_{\mathcal{O}} x$ implies that $\mathcal{O}(y)<\mathcal{O}(x)$, a contradiction. Therefore, either $x=y$ or x and y are not star twins, $y\in A(x)$, and $x\in A(y)$. Assume

x and y are not star twins, $y \in A(x)$, and $x \in A(y)$. By Lemma 5.1, $x \in A(y)$ implies that $B(y) \subseteq B(x)$ and $A(x) \subseteq A(y) \cup \{y\}$, and $y \in A(x)$ implies that $B(x) \subseteq B(y)$ and $A(y) \subseteq A(x) \cup \{x\}$. Therefore, $B(y) = B(x)$ and $A(x) \cup \{x\} = A(y) \cup \{y\}$, and so x and y are star twins, a contradiction. It follows that $x = y$, and so $\leq^w_{\mathcal{O}}$ is antisymmetric.

Finally, suppose $x \leq^w_{\mathcal{O}} y$ and $y \leq^w_{\mathcal{O}} z$ for distinct $x, y, z \in U$. Since $x \neq y$, it follows that $y \in A(x)$; similarly, since $y \neq z$, it follows that $z \in A(y)$. By Lemma 5.1, $A(y) \cup \{y\} \subseteq A(x) \cup \{x\}$, and $A(z) \cup \{z\} \subseteq A(y) \cup \{y\}$. Therefore, $z \in A(x)$. If x and z are not star twins, then $x \leq^w_{\mathcal{O}} z$, as desired, so we may assume that x and z are star twins and $A(z) \cup \{z\} = A(x) \cup \{x\}$. Then, $A(x) \cup \{x\} = A(y) \cup \{y\} = A(z) \cup \{z\}$. By Lemma 5.2, x and y are star twins and y and z are star twins. Now, $\mathcal{O}(x) < \mathcal{O}(y)$ and $\mathcal{O}(y) < \mathcal{O}(z)$, and so $\mathcal{O}(x) < \mathcal{O}(z)$. It follows that $x \leq^w_{\mathcal{O}} z$, and so $\leq^w_{\mathcal{O}}$ is transitive. □

Let G be a graph, let w be a normal weight function on G, let $\mathcal{O}$ be an ordering of $V(G)$, and let $U \subseteq V(G)$ be the set of w-unbalanced vertices of G. Let $X \subseteq U$. The *core of X*, denoted $\text{Core}_w(X)$, is the set of $\leq^w_{\mathcal{O}}$-minimal elements of X. By $\text{Core}_w(G)$ we denote the set of all $\leq^w_{\mathcal{O}}$-minimal elements of U; that is, core of the set of all w-unbalanced vertices of G. We typically assume that $\mathcal{O}$ is fixed in advance, and we omit the subscript w when the weight function is clear in context.

We need the following useful result, which is a natural corollary of the previous lemmas.

Lemma 5.4. *Let G be a graph and let w be a normal weight function on G. Let $x, y \in$ Core(G). Then, $x \notin A(y)$ and $y \notin A(x)$.*

Proof. Suppose that $x \in A(y)$. If x and y are not star twins, then $y \leq^w_{\mathcal{O}} x$ and so $x \notin \text{Core}(G)$, a contradiction. Therefore, x and y are star twins, and $y \in A(x)$. Assume up to symmetry between x and y that $\mathcal{O}(x) < \mathcal{O}(y)$. But now $x \leq^w_{\mathcal{O}} y$, so $y \notin \text{Core}(G)$, a contradiction. This completes the proof. □

Lemma 5.5. *Let G be a graph and let w be a normal weight function on G. Let $X \subseteq$ Core(G) be independent. Then, for every $x, x' \in X$, it holds that $x \in B(x')$ and $C(x) \cap A(x') = \emptyset$.*

Proof. By Lemma 5.4, $x \notin A(x')$, and since X is independent, it follows that $x \notin C(x')$. Therefore, $x \in B(x')$. Since $C(x) \subseteq N[x]$ and $A(x')$ is anticomplete to $B(x')$, it follows that $C(x) \cap A(x') = \emptyset$. □

Next, we prove a significant result regarding collections of canonical star separations.

Theorem 5.6. *Let G be a graph and let w be a normal weight function on G. Let $X \subseteq$ Core(G) be independent, and let $\mathcal{S} = \{S(x) \mid x \in X\}$ be the set of canonical star separations for the vertices of X. Let $q = \max_{S \in \mathcal{S}} |C(S)|$. Then, $\mathcal{S}$ is q-aligned.*

Proof. Let $S \in \mathcal{S}$ and let $x \in X$ be such that $S = S(x)$.

(4) *$C(S) \subseteq \beta_{\mathcal{S}}$ for all $S \in \mathcal{S}$.*

Let $S' \in \mathcal{S} \setminus \{S\}$ and let $y \in X \setminus \{x\}$ be such that $S' = S(y)$. By Lemma 5.4, it follows that $x \notin A(y)$. Since X is independent and y is complete to $C(y) \setminus \{y\}$, it follows $x \notin C(y)$. Therefore, $x \in B(y)$. Since x is complete to $C(x) \setminus \{x\}$ and $N(B(y)) \subseteq C(y)$, it follows that

$C(x) \subseteq B(y) \cup C(y)$. This holds for all $y \in X \setminus \{x\}$; thus $C(x) \subseteq \bigcap_{x \in X}(C(x) \cup B(x)) = \beta_{\mathcal{S}}$. This proves (4).

Since $x \in C(S) \subseteq N[x]$, it follows that $C(S)$ is connected, and by (4), $C(S) \cap \beta_{\mathcal{S}} = C(S)$, so condition (1i) holds. Let $\delta(S) = C(S)$. By the definition of q, it follows that $|C(S)| \leq q$ and $(A(S), C(S), B(S))$ is a separation of G, so condition (1ii) holds. Since x is w-unbalanced, it follows that $w(B(S)) > \frac{1}{2}$, and so $w(A(S)) < \frac{1}{2}$; thus condition (1iii) holds.

Finally, let D be a component of $\bigcup_{S \in \mathcal{S}} A(S) = V(G) \setminus \beta_{\mathcal{S}}$. Let $S \in \mathcal{S}$ be such that $D \cap A(S) \neq \emptyset$. By (4), $C(S) \subseteq \beta_{\mathcal{S}}$. Since $N(A(S)) \subseteq C(S)$, it follows that $D \subseteq A(S)$. Therefore, condition (2) holds. □

In view of Theorem 5.6, we also define a canonical anchor map for k-aligned collections of canonical star separations. Let $\mathcal{S}$ be a q-aligned sequence of canonical star separations. The canonical anchor map δ^* for $\mathcal{S}$ is defined such that for all $S(v) \in \mathcal{S}$, the anchor for $S(v)$ is $\delta^*(S) = v$. The canonical anchor map for collections of canonical star separations satisfies the property that δ^* is injective, which is convenient for arranging piece 3 of the central bag method.

We finish this section on star cutsets with an observation about central bags for certain collections of canonical star separations.

Lemma 5.7. *Let G be a graph and let w be a normal weight function on G. Let U be the w-unbalanced vertices of G, let $\mathcal{S} = \{S(v) \mid v \in \mathit{Core}(G)\}$, and let $\beta_{\mathcal{S}}$ be the central bag for $\mathcal{S}$. Then, $\beta_{\mathcal{S}} \cap U = \mathit{Core}(G)$.*

Proof. By the definitions of $\mathcal{S}$ and $\beta_{\mathcal{S}}$,

$$\beta_{\mathcal{S}} = \bigcap_{v \in \text{Core}(G)} (B(v) \cup C(v)).$$

First, we show that $\text{Core}(G) \subseteq \beta_{\mathcal{S}}$. Let $u \in \text{Core}(G)$. By Lemma 5.4, it follows that $u \in B(v) \cup C(v)$ for all $v \in \text{Core}(G)$. Therefore, $u \in \beta_{\mathcal{S}}$, and so $\text{Core}(G) \subseteq \beta_{\mathcal{S}}$.

Next, we show that $\beta_{\mathcal{S}} \cap U \subseteq \text{Core}(G)$. Let $u \in \beta_{\mathcal{S}} \cap U$ and suppose $u \notin \text{Core}(G)$. Then, there exists $v \in \text{Core}(G)$ such that $v \leq^w_{\mathcal{O}} u$, and so by the definition of $\leq^w_{\mathcal{O}}$, $u \in A(v)$. But $\beta_{\mathcal{S}} \subseteq B(v) \cup C(v)$, contradicting that $u \in \beta_{\mathcal{S}}$. Therefore, $\beta_{\mathcal{S}} \cap U \subseteq \text{Core}(G)$.

Since $\text{Core}(G) \subseteq \beta_{\mathcal{S}}$ and $\beta_{\mathcal{S}} \cap U \subseteq \text{Core}(G)$, it follows that $\beta_{\mathcal{S}} \cap U = \text{Core}(G)$. □

6. Revisions of collections of separations

In this section, we explain how to revise certain collections of separations to create a k-aligned collection of separations. The purpose of the revision is to deal with collections of separations $\mathcal{S}$ such that $\mathcal{S}$ is almost k-aligned, but fails to satisfy condition (1i): $C(S) \cap \beta_{\mathcal{S}}$ is not necessarily connected for all $S \in \mathcal{S}$.

Let G be a graph, let w be a normal weight function on G, and assume that G does not have a w-balanced separator of size at most k. Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be an A-laminar collection of separations with $|C(S)| \leq k$ for all $S \in \mathcal{S}$. For each $S_i = (A_i, C_i, B_i)$, since C_i is not a w-balanced separator, it holds that either $w(A_i) > 1/2$ or $w(B_i) > 1/2$; assume

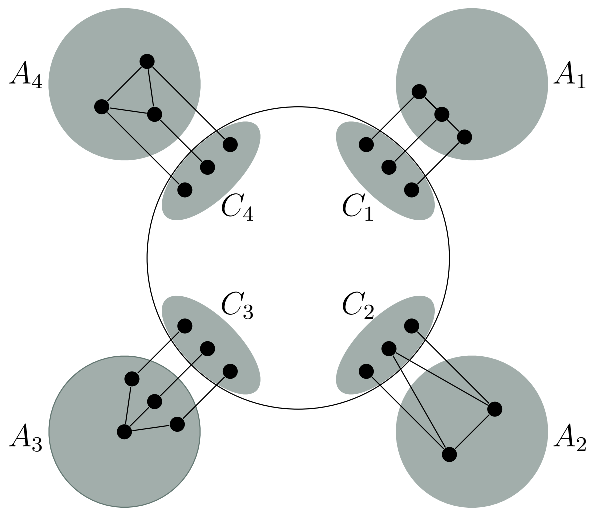

FIGURE 7. An illustration of the central bag for a revision of a laminar collection of separations. The central bag includes the center circle, plus the drawn vertices in A_1, A_2, A_3, and A_4.

that $w(A_i) < \frac{1}{2}$ and $w(B_i) > \frac{1}{2}$, following our usual convention. We also assume that $\mathcal{S}$ is shield-minimal. Finally, we assume that for each $S_i = (A_i, C_i, B_i)$, it holds that $G[A_i \cup C_i]$ is connected.

For every $1 \leq i \leq m$, if $C_i \neq C_j$ for all $j < i$, let $X_i \subsetneq A_i$ be inclusion-wise minimal such that $X_i \cup C_i$ is connected; otherwise, let $X_i = \emptyset$. Let $S_i' = (A_i \setminus X_i, X_i \cup C_i, B_i)$. Let $X(S_i) \coloneqq X_i$. Under these conditions, we call $\mathcal{S}' = \{S_1', \dots, S_m'\}$ a *revision* of $\mathcal{S}$. Note that revisions of $\mathcal{S}$ are not necessarily unique, since the choice of X_i may not be unique.

Lemma 6.1. *Let G be a graph, let w be a normal weight function on G, and assume that G does not have a w-balanced separator of size at most k. Let $\mathcal{S}$ be an A-laminar collection of separations of G such that $|C(S)| \leq k$ for all $S \in \mathcal{S}$ and let $\mathcal{S}'$ be a revision of $\mathcal{S}$. Then, $\mathcal{S}'$ is k-aligned.*

Proof. Let $\mathcal{S} = \{S_1, \dots, S_m\}$. First, we show:

(5) *For all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, m\} \setminus \{i\}$, it holds that $C(S_i') \subseteq C(S_j') \cup B(S_j')$.*

Since $\mathcal{S}$ is A-laminar, it holds that $C(S_i) \cup A(S_i) \subseteq C(S_j) \cup B(S_j)$. By construction, $X_i \subsetneq A(S_i)$ and $C(S_i') = C(S_i) \cup X_i$. Therefore, $C(S_i') \subseteq C(S_j) \cup B(S_j)$. Finally, $C(S_j) \subseteq C(S_j')$ and $B(S_j) = B(S_j')$, so $C(S_i') \subseteq C(S_j') \cup B(S_j')$. This proves (5).

Let $S_i' \in \mathcal{S}'$. By (5), it follows that $C(S_i') \subseteq \bigcap_{S' \in \mathcal{S}'} B(S') \cup C(S')$, so $C(S_i') \cap \beta_{\mathcal{S}'} = C(S_i')$. By construction, $C(S_i')$ is connected. Therefore, condition (1i) holds. Condition (1ii) holds with $\delta(S_i') = C(S_i)$. Since $A(S_i') \cup (C(S_i') \setminus \delta(S_i')) = A(S_i)$ and $w(A(S_i)) < \frac{1}{2}$, condition (1iii) holds.

Finally, we show that $\mathcal{S}'$ is A-laminar. Let $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, m\} \setminus \{i\}$. By (5), it follows that $C(S_i') \cap A(S_j') = \emptyset$. Since $\mathcal{S}$ is A-laminar, it follows that $A(S_i) \cap A(S_j) = \emptyset$.

By construction, $A(S_i') \subseteq A(S_i)$ and $A(S_j') \subseteq A(S_j)$, so $A(S_i') \cap A(S_j') = \emptyset$. Therefore, $\mathcal{S}'$ is A-laminar, and thus, by Lemma 4.1, condition (2) of the definition of k-aligned holds. This completes the proof. □

The next lemma states a relationship between the central bag for $\mathcal{S}$ and the central bag for a revision $\mathcal{S}'$ of $\mathcal{S}$.

Lemma 6.2. *Let G be a graph, let w be a normal weight function on G, and assume that G does not have a w-balanced separator of size at most k. Let $\mathcal{S}$ be an A-laminar collection of separations of G such that $|C(S)| \leq k$ for all $S \in \mathcal{S}$ and let $\mathcal{S}'$ be a revision of $\mathcal{S}$. Then, $\beta_{\mathcal{S}'} = \beta_{\mathcal{S}} \cup \bigcup_{S' \in \mathcal{S}'} X(S')$.*

Proof. For every $S_i \in \mathcal{S}$, it holds that $\beta_{\mathcal{S}} \subseteq B(S_i) \cup C(S_i) \subseteq B(S_i') \cup C(S_i')$, and so $\beta_{\mathcal{S}} \subseteq \beta_{\mathcal{S}'}$. Also, since $\mathcal{S}$ is A-laminar and $X(S_i') \subseteq A(S_i)$, it follows that $X(S_i') \subseteq B(S') \cup C(S')$ for all $S' \in \mathcal{S}'$. Therefore, $\bigcup_{S' \in \mathcal{S}'} X(S') \subseteq \beta_{\mathcal{S}'}$.

Recall that $\beta_{\mathcal{S}}' = \bigcap_{S' \in \mathcal{S}'} (B(S') \cup C(S'))$. Let $x \in \beta_{\mathcal{S}'} \setminus \beta_{\mathcal{S}}$. It follows that there exists $S \in \mathcal{S}$ such that $x \notin B(S) \cup C(S)$. Therefore, $x \in X(S)$, and so $\beta_{\mathcal{S}'} \setminus \beta_{\mathcal{S}} \subseteq \bigcup_{S \in \mathcal{S}'} X(S')$. This completes the proof. □

We also prove the following useful lemma.

Lemma 6.3. *Let G be a graph and let $S = (A, C, B)$ be a separation of G such that $|C| \leq k$. Assume that $A \cup C$ is connected and let $X \subseteq A$ be inclusion-wise minimal such that $X \cup C$ is connected. Then, for all $u \in X \cup C$, it holds that $|N(u) \cap X| \leq k$.*

Proof. Let $u \in X \cup C$ and suppose for the sake of contradiction that $\{x_1, \ldots, x_{k+1}\} \subseteq N(u) \cap X$. Since X is inclusion-wise minimal such that $X \cup C$ is connected, it follows that $X \setminus \{x_i\} \cup C$ is not connected for all $i \in \{1, \ldots, k+1\}$. Let $c_i \in C$ be a vertex of C such that there is no path from u to c_i in $G[X \setminus \{x_i\} \cup C]$ for $i \in \{1, \ldots, k+1\}$. Since $|C| \leq k$, it follows that there exist $1 \leq i < j \leq k+1$ such that $c_i = c_j$. Let P_i be a path from u to c_i in $G[X \cup C]$. Because P_i is not a path in $G[X \setminus \{x_i\} \cup C]$, it follows that x_i is the unique neighbor of u in P_i. But now P_i is a path from u to c_j in $G[X \setminus \{x_j\} \cup C]$, a contradiction. This completes the proof. □

7. Tree decompositions and k-blocks

One neat application of the method described in the previous subsection is to prove a relationship between treewidth and a graph substructure called a k-block. Let G be a graph. A *k-block of G* is a set $X \subseteq V(G)$ such that $|X| \geq k$ and for every pair $x_1, x_2 \in X$ and every set $C \subseteq V(G) \setminus \{x_1, x_2\}$ with $|C| < k$, it holds that x_1 and x_2 are in the same connected component of $G \setminus C$.

The largest k such that there exists a k-block in a graph G is an easy lower bound for the treewidth of G. First we need the following:

Proposition 7.1. *Let G be a graph and let (T, χ) be a tree decomposition of G. Let $X \subseteq V(G)$ be such that for every $x_1, x_2 \in X$, there exists $t_{x_1 x_2} \in V(T)$ where $\{x_1, x_2\} \subseteq \chi(t_{x_1 x_2})$. Then, there exists $t \in V(T)$ such that $X \subseteq \chi(t)$.*

Proposition 7.2. *Let G be a graph and let k be a positive integer. Suppose $X \subseteq V(G)$ is a k-block of G. Then,* $\mathrm{tw}(G) \geq k-1$.

Proof. Let (T, χ) be a tree decomposition of G. If there exists $t \in V(T)$ such that $X \subseteq \chi(t)$, then $\mathrm{tw}(G) \geq |X| - 1 = k-1$, so we may assume that there does not exist $t \in V(T)$ such that $X \subseteq \chi(t)$. By Proposition 7.1, there exist $x_1, x_2 \in X$ such that $\{x_1, x_2\} \not\subseteq \chi(t)$ for all $t \in \chi(t)$. Let $t_1, t_2 \in V(T)$ be such that $x_1 \in \chi(t_1)$, $x_2 \in \chi(t_2)$, and t_1 and t_2 are the closest pair of vertices of T with this property. Let t_3 be the vertex adjacent to t_1 on the path from t_1 to t_2 in T, so $x_1 \notin \chi(t_3)$ and $x_2 \notin \chi(t_3)$. Then, x_1 and x_2 are in different components of $G \setminus (\chi(t_1) \cap \chi(t_3))$. By the definition of k-block, it follows that $|\chi(t_1) \cap \chi(t_3)| \geq k$, and thus that $|\chi(t_1)| \geq k$. It follows that the width of (T, χ) is at least $k-1$ for all tree decompositions (T, χ) of G, so $\mathrm{tw}(G) \geq k-1$. □

Proposition 7.2 shows that the presence of large k-blocks in a graph forces large treewidth. Conversely, the absence of k-blocks in a graph leads to the existence of a special type of tree decomposition. In [13], the following is proven.

Theorem 7.3 ([13], Theorem 1)**.** *Let G be a graph and let k be a positive integer. Suppose that G has no $(k+1)$-block. Then, G has a tree decomposition (T, χ) such that (T, χ) has adhesion less than k and every torso of (T, χ) has at most k vertices of degree at least $2k(k-1)$.*

Theorem 7.3 allows us to make a reduction between the treewidth of graphs of bounded degree and the treewidth of graphs with no k-block. Specifically, we can prove the following theorem:

Theorem 7.4. *Let $\mathcal{G}$ be a hereditary graph class. Suppose that every graph $G \in \mathcal{G}$ of maximum degree Δ has $\mathrm{tw}(G) \leq f(\Delta)$, where $f : \mathbb{Z} \to \mathbb{Z}$. Let $G \in \mathcal{G}$ and $k > 0$ be such that G has no $(k+1)$-block. Then,*

$$\mathrm{tw}(G) \leq 2f((2^{2k(k-1)} - 1)(1+k))(2k^2(k-1)+1) + 2k + 2.$$

Proof. Suppose for a contradiction that $\mathrm{tw}(G) > 2f((2^{2k(k-1)} - 1)(1+k))(2k^2(k-1)+1) + 2k+2$. By Lemma 2.4, there exists a normal weight function w on G such that G does not have a w-balanced separator of size at most $f((2^{2k(k-1)} - 1)(1+k))(2k^2(k-1)+1) + k + 1$. Since $f((2^{2k(k-1)}-1)(1+k))(2k^2(k-1)+1) \geq 0$, it follows that G does not have a w-balanced separator of size at most k. Let (T, χ) be the tree decomposition of G given by Theorem 7.3, so (T, χ) has adhesion less than k and every torso of (T, χ) has at most k vertices of degree at least $2k(k-1)$. Let $\mathcal{T}$ be the laminar collection of separations corresponding to (T, χ), as in Theorem 2.6. Because (T, χ) has adhesion less than k, it follows that $|C(S)| < k$ for all $S \in \mathcal{T}$. Since G does not have a w-balanced separator of size at most k, it follows that (T, χ) is w-unbalanced. Let $t \in V(T)$ be the w-heavy vertex of T as in Lemma 2.2. Let $\mathcal{S} \subseteq \mathcal{T}$ be the separations of $\mathcal{T}$ corresponding to edges of T incident with t. Let $\mathcal{S}'$ be a revision of $\mathcal{S}$ and let $\beta_{\mathcal{S}'}$ be the central bag for $\mathcal{S}'$. By Lemma 6.2 and Theorem 4.5, $\beta_{\mathcal{S}'} = \beta_{\mathcal{S}} \cup \bigcup_{S' \in \mathcal{S}'} X(S') = \chi(t) \cup \bigcup_{S' \in \mathcal{S}'} X(S')$.

Let $F \subseteq \chi(t)$ be the vertices of $\chi(t)$ of degree at least $2k(k-1)$ in the torso of $\chi(t)$. By Theorem 7.3, $|F| \leq k$. First, we claim that $\beta_{\mathcal{S}'} \setminus F$ has maximum degree at most $2k(k^2-1)$.

By the choice of F, it follows that $\chi(t) \setminus F$ has maximum degree $2k(k-1)$. For every vertex $v \in \chi(t)$, if $v \in C(S)$ for some $S \in \mathcal{S}$, then v is complete to $C(S)$ in the torso of $\chi(t)$. Therefore, every vertex of $\chi(t) \setminus F$ is in $C(S)$ for at most

$$\sum_{i=1}^{2k(k-1)} \binom{2k(k-1)}{i} = 2^{2k(k-1)} - 1$$

separations $S \in \mathcal{S}$. Finally, by Lemma 6.3, since $N(X(S'_i)) \subseteq C(S_i)$ for all $S'_i \in \mathcal{S}'$, and since $|C(S_i)| \leq k$ for all $S'_i \in \mathcal{S}'$, it follows that the maximum degree of $\beta_{\mathcal{S}'} \setminus F$ is $(2^{2k(k-1)}-1)(1+k)$.

Since $\beta_{\mathcal{S}'} \setminus F$ has maximum degree $(2^{2k(k-1)} - 1)(1 + k)$, it follows from the assumptions of the theorem that $\text{tw}(\beta_{\mathcal{S}'} \setminus F) \leq f((2^{2k(k-1)} - 1)(1 + k))$. We want to show that $\beta_{\mathcal{S}'}$ has a $w_{\mathcal{S}'}$-balanced separator of bounded size. We may assume that $w_{\mathcal{S}'}(F) < 1/2$, otherwise F is a $w_{\mathcal{S}'}$-balanced separator of $\beta_{\mathcal{S}'}$ of bounded size. Let $\overline{w} : \beta'_{\mathcal{S}} \setminus F \to [0, 1]$ be given by restricting $w_{\mathcal{S}'}$ to $\beta_{\mathcal{S}'} \setminus F$ and normalizing; that is,

$$\overline{w}(v) = \frac{w_{\mathcal{S}'}(v)}{1 - w_{\mathcal{S}'}(F)}$$

for all $v \in \beta_{\mathcal{S}'} \setminus F$. By Lemma 2.3, $\beta_{\mathcal{S}'} \setminus F$ has a $\overline{w}$-balanced separator X with $|X| \leq f((2^{2k(k-1)}-1)(1+k))+1$. Let $Y = X \cup F$. Now, the components of $(\beta_{\mathcal{S}'} \setminus F) \setminus X$ are the same as the components of $\beta_{\mathcal{S}'} \setminus Y$. Let D be such a component. Since $0 \leq w_{\mathcal{S}'}(F) < 1/2$, it follows that $\overline{w}(D) \geq w_{\mathcal{S}'}(v)$. Since X is a $\overline{w}$-balanced separator of $\beta_{\mathcal{S}'} \setminus F$, it follows that $\overline{w}(D) \leq 1/2$, and so $w_{\mathcal{S}'}(D) \leq 1/2$. It follows that Y is a $w_{\mathcal{S}'}$-balanced separator of $\beta_{\mathcal{S}'}$ of size at most $f((2^{2k(k-1)}-1)(1+k))+k+1$. Finally, we apply Theorem 4.4. Suppose that $S \in \mathcal{S}'$ crosses Y. Let $\mathcal{S}'_F$ be the set of separations $S' \in \mathcal{S}'$ such that $C(S') \subseteq F$. Let $\mathcal{S}'' = \mathcal{S}' \setminus \mathcal{S}'_F$. Since $C(S)$ is connected, it follows that $C(S) \cap Y \neq \emptyset$ and that $C(S) \not\subseteq Y$, so $S \in \mathcal{S}''$ and $C(S) \cap X \neq \emptyset$. Every vertex in X is in at most $2k(k-1)$ separations of $\mathcal{S}''$, so it follows that the number of separations of $\mathcal{S}'$ that cross Y is at most $|X| \cdot 2k(k-1) = f((2^{2k(k-1)} - 1)(1+k))(2k(k-1))$.

Now, by Theorem 4.4, G has a w-balanced separator of size $(f((2^{2k(k-1)} - 1)(1 + k)) + k + 1) + f((2^{2k(k-1)} - 1)(1 + k))(2k(k - 1) \cdot k = f((2^{2k(k-1)} - 1)(1 + k))(2k^2(k - 1) + 1) + k + 1$, a contradiction. This completes the proof. □

Theorem 7.4 is intriguing in its own right: it reveals that from the point of view of having bounded treewidth, for hereditary graph classes, the condition of having bounded degree is equivalent to the condition of having no k-block. Let $\mathcal{G}$ be a hereditary graph class. If $G \in \mathcal{G}$ has maximum degree Δ, then G has no $(\Delta + 1)$-block. Therefore, if graphs in $\mathcal{G}$ with no k-block have bounded treewidth, then graphs in $\mathcal{G}$ with bounded degree have bounded treewidth. Theorem 7.4 shows that this is a necessary and sufficient relationship: graphs in $\mathcal{G}$ with no k-block have bounded treewidth if and only if graphs in $\mathcal{G}$ with bounded degree have bounded treewidth.

References

[1] Abrishami, T., Alecu, B., Chudnovsky, M., Hajebi, S., Spirkl, S., and Vušković, K. Induced subgraphs and tree decompositions V. One neighbor in a hole. *arXiv* (2022).

[2] Abrishami, T., Chudnovsky, M., Dibek, C., Hajebi, S., Rzazewski, P., Spirkl, S., and VuÅąkoviÄĞ, K. Induced subgraphs and tree decompositions II. Toward walls and their line graphs in graphs of bounded degree. *arXiv: arXiv:2108.01162* (2021).

[3] Abrishami, T., Chudnovsky, M., Dibek, C., and Rzazewski, P. Polynomial-time algorithm for maximum independent set in bounded-degree graphs with no long induced claws. *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022* (2022), 1448âĂŞ1470.

[4] Abrishami, T., Chudnovsky, M., Dibek, C., and VuÅąkoviÄĞ, K. Submodular functions and perfect graphs. *arXiv: arXiv:2110.00108* (2021).

[5] Abrishami, T., Chudnovsky, M., Hajebi, S., and Spirkl, S. Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth. *Advances in Combinatorics 6* (2022).

[6] Abrishami, T., Chudnovsky, M., Hajebi, S., and Spirkl, S. Induced subgraphs and tree decompositions IV. (Even hole, diamond, pyramid)-free graphs. *Electronic Journal of Combinatorics 30*, 2 (2023).

[7] Abrishami, T., Chudnovsky, M., and Vušković, K. Induced subgraphs and tree decompositions I. Even-hole-free graphs of bounded degree. *Journal of Combinatorial Theory, Series B 157* (2022), 144–175.

[8] Harvey, D. J., and Wood, D. R. Parameters tied to treewidth. *Journal of Graph Theory 84*, 4 (2017), 364–385.

[9] Robertson, N., and Seymour, P. D. Graph minors II. Algorithmic aspects of tree-width. *Journal of Algorithms 7* (1986), 309–322.

[10] Robertson, N., and Seymour, P. D. Graph minors X. Obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B 52*, 2 (1991), 153–190.

[11] Trotignon, N. Perfect graphs: a survey. In *Topics in Chromatic Graph Theory*. Cambridge University press, 2015, pp. 137–160.

[12] Vušković, K. Even-hole-free graphs: a survey. *Applicable Analysis and Discrete Mathematics 4*, 2 (2010), 219 – 240.

[13] Weissauer, D. On the block number of graphs. *SIAM Journal on Discrete Mathematics 33*, 1 (2019).