Maximum weight induced subgraphs of bounded treewidth and the container method

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Introduction

An independent set in a graph G is a set $I \subseteq V(G)$ such that no edge in $E(G)$ has both endpoints in I.

Maximum weight independent set problem: find a maximum weight independent set in polynomial time

Treewidth

A tree decomposition of a graph G is (T, β), where T is a tree and $\beta : V(T) \to 2^{V(G)}$ is a map from vertices of T to subsets of $V(G)$, such that

- $\bigcup_{t \in V(T)} \beta(t) = V(G)$,
- for all $v_1 v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \beta(t)$, and
- for all $v \in V(G)$, the set $\{t \in V(T) : v \in \beta(t)\}$ induces a connected subtree of T

The width of (T, β) is $\max_{v \in V(T)} |\beta(t)| - 1$. The treewidth of a graph G is the minimum width of a tree decomposition of G.

Treewidth

MAXIMUM WEIGHT INDEPENDENT SET is an instance of MAXIMUM WEIGHT INDUCED SUBGRAPH OF BOUNDED TREEWIDTH:

- Independent sets have treewidth 0. If I is an independent set, (T, β) is a tree decomposition of I, where $V(T) = V(I)$ and $\beta(t) = t$ for all $t \in V(T)$

Definitions

A graph G is chordal if every cycle has a chord. A chordal completion of G is a graph $G + F$, where $F \subseteq \binom{V(G)}{2} \setminus E(G)$, such that $G + F$ is chordal. A chordal completion $G + F$ is minimal if $G + F'$ is not chordal for any $F' \subsetneq F$

A potential maximal clique (PMC) of G is a maximal clique of a minimal chordal completion $G + F$.

Tree decompositions and PMCs

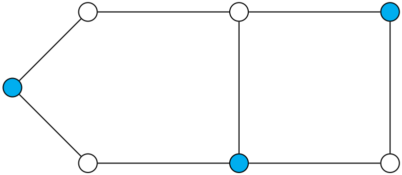
A graph G is chordal if and only if there exists a tree decomposition (T, β) of G such that every bag is a maximal clique in G. If G is chordal, such a tree decomposition is called a clique tree of G.

Lemma

If $G + F$ is a minimal chordal completion of G and (T, β) is a clique tree of $G + F$, then (T, β) is a tree decomposition of G where every bag is a PMC of G.

I-good Chordal Completions

Let G be a graph and let I be an independent set of G. A minimal chordal completion $G + F$ is I-good if $e \cap I = \emptyset$ for all $e \in F$.



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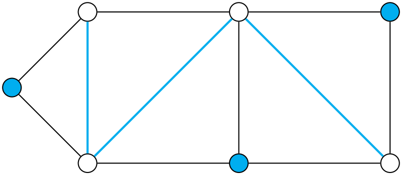

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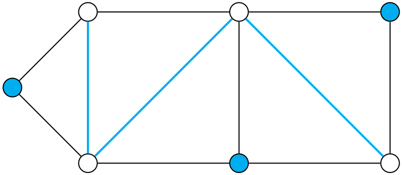


Maximum weight independent set in an I-good minimal chordal completion of $G \iff$ maximum weight independent set in G

I-good Chordal Completions

Lemma

Let G be a graph. For every independent set I of G, there exists an I-good minimal chordal completion of G.

Proof.

Let I be an independent set of G and let $G + F'$ be the graph given by turning $V(G) \setminus I$ into a clique. Then, $G + F'$ is chordal. It follows that there exists $F \subseteq F'$ such that $G + F$ is an I-good minimal chordal completion. $\square$

Theorem (Bouchitte, Todinca)

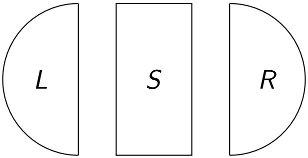
Given a list Π of all PMCs of G, one can find a maximum weight independent set of G in time polynomial in $|\Pi|$ and $|V(G)|$.

Corollary

If G has polynomially many PMCs, then one can find a maximum weight independent set in G in polynomial time.

Minimal separators

A minimal separator of a graph G is a set $S \subseteq V(G)$ such that there exist two connected components L, R of $V(G) \setminus S$ with $N(L) = N(R) = S$.



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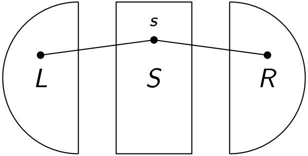


Every $s \in S$ has a neighbor in L and a neighbor in R.

Minimal separators and PMCs

Lemma

A graph G has polynomially many minimal separators if and only if G has polynomially many potential maximal cliques.

Lemma

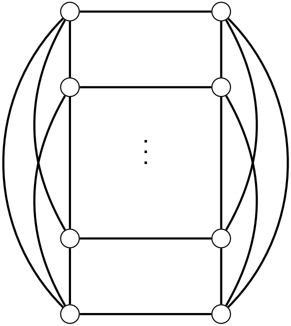
The minimal separators of a graph G can be listed in time polynomial in the number of minimal separators of G.

Lemma

Given a list $\mathcal{S}$ of all minimal separators of G, the potential maximal cliques of G can be listed in time polynomial in $\mathcal{S}$.

Exponentially many minimal separators

k-prism has $2^k - 2$ minimal separators:



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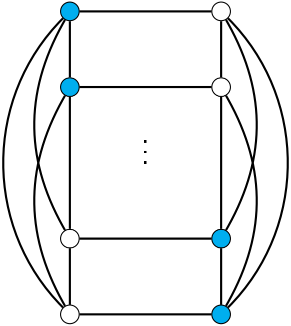

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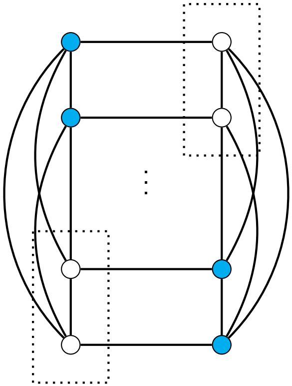

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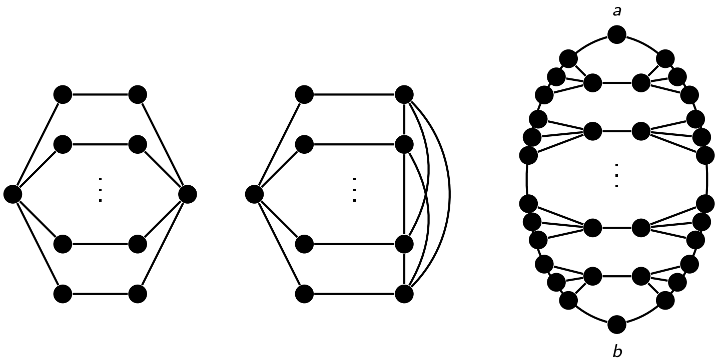
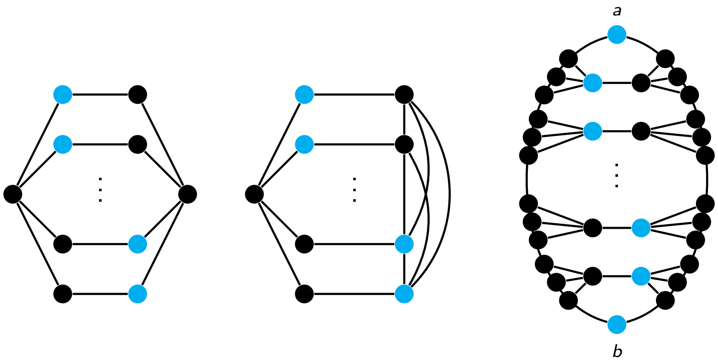


k-theta, k-pyramid, k-turtle

Exponentially many minimal separators



k-theta, k-pyramid, k-turtle

Exponentially many minimal separators

Theorem (A., Chudnovsky, Dibek, Thomassé, Trotignon, Vušković)
If G is (theta, pyramid, prism, turtle)-free, then G has polynomially many minimal separators.

The PMC method

Theorem (Lokshtanov, Vatshelle, Villanger)

Given a list Π of vertex sets of G, one can find in time polynomial in $|\Pi|$ and $|V(G)|$ a maximum weight independent set I such that there exists a tree decomposition (T, β) of G such that $\beta(v) \in \Pi$ and $|\beta(v) \cap I| \leq 1$ for all $v \in V(T)$.

Method: Need to find a polynomial-size list Π of PMCs of G such that for a maximum independent set I of G, every PMC of some I-good minimal chordal completion is in Π

P_5-free graphs

Theorem (Lokshtanov, Vatshelle, Villanger)

Given a P_5-free graph G, one can compute in polynomial time a polynomial-size list Π of vertex sets of G such that for every maximal independent set I of G, there exists an I-good minimal chordal completion $G + F$ of G such that every maximal clique of $G + F$ is in Π.

P_6-free graphs

Theorem (Grzesik, Klimošová, Pilipczuk, Pilipczuk)
Given a P_6-free graph G, one can compute in polynomial time a polynomial-size list Π of vertex sets of G such that for every maximal independent set I of G, there exists an I-good minimal chordal completion $G + F$ of G such that every maximal clique of $G + F$ is in Π.

Containers

Let F be an induced subgraph of G. An F-container of a set $C \subseteq V(G)$ is a set $A \subseteq V(G)$ such that $C \subseteq A$ and $A \cap F = C \cap F$.

Idea: Find I-containers of minimal separators and potential maximal cliques of G.

Theorem

Suppose for a graph G, we are given a polynomial-size set Π of subsets of $V(G)$ such that for every independent set I of G and every PMC Ω of G, if $|V(I) \cap \Omega| \leq 1$, then Π has an I-container for Ω. Then, one can in polynomial time find a maximum weight independent set of G.

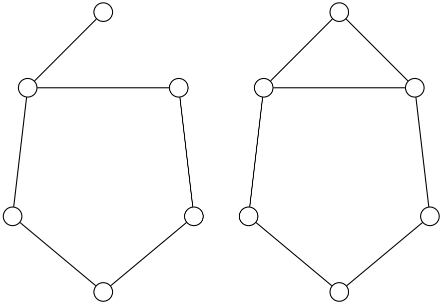
The PMC container method

Theorem

Suppose for a graph G and an integer $k \geq 0$, we are given a polynomial-size set Π of subsets of $V(G)$ such that for every induced subgraph F of G of treewidth less than k and every PMC Ω of G, if $|V(F) \cap \Omega| \leq k$, then Π has an F-container for Ω. Then, one can in polynomial time find a maximum weight induced subgraph of G of treewidth less than k.

Results

Let $\mathcal{C}$ be the class of graphs with no hole of length greater than 5 and no extended C_5 as an induced subgraph.



Extended C_5

Theorem

Given a graph $G \in \mathcal{C}$ and an integer k, one can in time $n^{\mathcal{O}(k)}$ compute a list X of polynomial size such that for every induced subgraph F of treewidth less than k and every potential maximal clique Ω of G, there exists $S \in X$ such that S is an F-container for Ω.

Maximum weight independent set in long-hole-free graphs can be solved in polynomial time.

The End

Questions?