Induced subgraph obstructions to bounded treewidth

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Seminar, TU Ilmenau
May 2021

A **tree decomposition** (T, χ) of a graph G is a tree T and a map $\chi : V(T) \to 2^{V(G)}$, such that

(i) for all $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$

(ii) for all $v_1 v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$

(iii) for all $v \in V(G)$, the set $\{t \in V(T) : v \in \chi(t)\}$ induces a connected subtree of T

The **width** of (T, χ) is $\max_{t \in V(T)} |\chi(t)| - 1$.

The **treewidth** of G is the minimum width of a tree decomposition of G.

Introduction

Bounded treewidth $\rightarrow$ efficient algorithms

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Which graphs have bounded treewidth?

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Look at “substructure obstructions”

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Grid Minor Theorem (Robertson and Seymour, '91)

If $\text{tw}(G) > f(k)$, then G contains a $k \times k$ grid as a minor.

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Theorem (Fomin, Golovach, Thilikos, '11)

If $tw(G) > f(k)$*, then* G *contains one of the following graphs as a contraction.*

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Induced subgraph obstructions?

Theorem (Sintiari and Trotignon, '20)

For all k there exist graphs G with no K_4 and no even hole such that $tw(G) > k$.

Question

What are the induced subgraph obstructions to bounded treewidth in graphs with maximum degree δ?

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Conjecture (Aboulker, Adler, Kim, Sintiari, Trotignon, '20)

If G has maximum degree δ and no $W_{k\times k}$ or $L(W_{k\times k})$, then $\text{tw}(G) < f(k,\delta)$.

Results (Preview)

Theorem 1 (A., Chudnovsky, Vuskovic, '20)

Even-hole-free graphs with bounded degree have bounded treewidth.

Theorem 2 (A., Chudnovsky, Dibek, Rzazewski, '21)

If G has maximum degree δ, no $S_{t,t,t}$, and no $L(Sub(W_{k\times k}))$, then $\text{tw}(G) < f(\delta, t, k)$.

Theorem 3 (A., Chudnovsky, Dibek, Hajebi, Spirkl, Vuskovic, '21)

If G has maximum degree δ, no t-theta, no t-pyramid, and no $L(Sub(W_{k\times k}))$, then $\text{tw}(G) < f(\delta, t, k)$.

Theorem 4 (A., Chudnovsky, Dibek, Hajebi, Spirkl, '21)

Let T be a subcubic caterpillar with b branch vertices. If G has maximum degree δ and no T or $L(T)$, then $\text{tw}(G) < f(\delta, b)$.

Balanced separators

Treewidth and balanced separators

Theorem (Harvey and Wood)

If G has a (w, c)-balanced separator of size k for every $w : V(G) \to [0, 1]$ such that there exists $S \subseteq V(G)$ such that $w(v) = \frac{1}{|S|}$ if $v \in S$ and $w(v) = 0$ otherwise, then $tw(G) < \frac{k}{1-c}$.

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Theorem (Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, Saurabh)

If $tw(G) \leq k$, then G has a $(w, \frac{1}{2})$-balanced separator of size $k+1$ for all weight functions w.

Balanced separators

G has no d-**bounded** (w, c)-**balanced separator**:

Separations

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Two separations (A_1, C_1, B_1) and (A_2, C_2, B_2) are **non-crossing** if, up to symmetry, $A_1 \cup C_1 \subseteq B_2 \cup C_2$ and $A_2 \cup C_2 \subseteq B_1 \cup C_1$.

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A_2	$\varnothing$	$\varnothing$	
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A collection of separations $\mathcal{S}$ is **laminar** if for every $S_1, S_2 \in \mathcal{S}$, S_1 and S_2 are non-crossing

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There is an equivalence between laminar collections of separations of G and tree decompositions of G.

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Central bags

If $\mathcal{S}$ is loosely laminar, the **central bag** for $\mathcal{S}$, $\beta_{\mathcal{S}}$, is:

$$\beta_{\mathcal{S}} = \bigcap_{S \in \mathcal{S}} B(S) \cup C(S).$$

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If Y is a $(d - t)$-bounded $(w_{\mathcal{S}}, c)$-balanced separator of $\beta_{\mathcal{S}}$, then $Y' = N^t[Y]$ is a d-bounded (w, c)-balanced separator of G.

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Separation dimension

The **dimension** of $\mathcal{S}$ is the minimum k such that there exists a partition of $\mathcal{S}$ into k collections $\mathcal{S}_1, \ldots, \mathcal{S}_k$, such that every $\mathcal{S}_i$ is loosely laminar

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Central bag for a collection $\mathcal{S}$ of dimension k:

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Central bag method: Given a collection $\mathcal{S}$ of separations, find an induced subgraph $\beta_{\mathcal{S}}$ of G such that if G has unbounded treewidth, then $\beta_{\mathcal{S}}$ has unbounded treewidth

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Want: a collection $\mathcal{S}$ so that finding the treewidth of $\beta_{\mathcal{S}}$ is easier than finding the treewidth of G

Canonical separations

For $X \subseteq V(G)$, the canonical separation for X, $S_X = (A_X, C_X, B_X)$, is:

- B_X: largest weight component of $G \setminus N[X]$
- C_X: $X \cup (N(X) \cap N(B_X))$
- A_X: $V(G) \setminus (B_X \cup C_X)$

Forcers

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Lemma

If X breaks Y, then $Y \cap A_X \neq \emptyset$.

Forcers

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Lemma

The central bag for $\mathcal{S}_X$ is F-free for every graph F such that F is an X-forcer for G.

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The central bag $\beta_{\mathcal{S}_X}$ is F-free for every X-forcer F

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- Subdivided claws are forcers
- Base case: claw-free graphs

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If G has maximum degree δ, no t-theta, no t-pyramid, and no $L(Sub(W_{k \times k}))$, then $\text{tw}(G) < f(\delta, t, k)$.

Corollary

If G is (theta, triangle)-free and has maximum degree δ, then $\text{tw}(G) < f(\delta)$.

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