Maximum weight induced subgraphs of bounded treewidth and the container method

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Introduction

An **independent set** in a graph G is a set $I \subseteq V(G)$ such that no edge in $E(G)$ has both endpoints in I.

Maximum independent set (MIS): find a maximum independent set in a graph G

[Karp 1972]: MIS is NP-hard

Introduction

An **induced subgraph** of G is a subgraph of G formed by vertex deletions

If G, H are graphs, G is **H-free** if G does not contain H as an induced subgraph

If G is a graph and $\mathcal{H}$ is a set of graphs, G is **$\mathcal{H}$-free** if G is H-free for all $H \in \mathcal{H}$

Question: for which graphs H is MIS solvable in polynomial time in H-free graphs? For which sets of graphs $\mathcal{H}$ is MIS solvable in polynomial time in $\mathcal{H}$-free graphs?

Introduction

A **hole** is an induced cycle of length at least 4

P_k: path on k vertices

perfect graphs: graphs with no odd hole or odd antihole (complement of an odd hole)

long-hole-free graphs: graphs with no hole of length five or greater

MIS in H-free and $\mathcal{H}$-free graphs

MIS is solvable in polynomial time in:

- perfect graphs (Grötschel, Lovász, and Schrijver 1981)
- claw-free graphs (Sbihi, Minty, 1980) and fork-free graphs (Alekseev, 2004; Lozin and Milanic, 2006)
- P_5-free graphs (Lokshtanov, Vatshelle, and Villanger 2013)
- P_6-free graphs (Grzesik, Klimošová, Pilipczuk, and Pilipczuk, 2017)

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Theorem (A, Chudnovsky, Rzążewski, Pilipczuk, Seymour):
MIS is solvable in polynomial time in long-hole-free graphs

Definitions

A graph G is **chordal** if every cycle has a chord. A **chordal completion** of G is a set $F \subseteq \binom{V(G)}{2} \setminus E(G)$ such that $G + F$ is chordal. A chordal completion F is **minimal** if $G + F'$ is not chordal for any $F' \subsetneq F$

A set Ω is a **potential maximal clique (PMC)** of G if there exists a minimal chordal completion F of G such that Ω is a maximal clique of $G + F$.

The PMC method

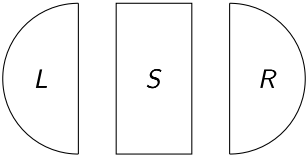
Theorem (Fomin, Villanger)
Given a list Π of all PMCs of G, one can find a maximum weight independent set of G in time polynomial in $|\Pi|$ and $|V(G)|$.

Corollary
If G has polynomially many PMCs, then one can find a maximum weight independent set in G in polynomial time.

Key idea: PMCs are used in dynamic programming to find a maximum independent set.

Minimal separators

A **minimal separator** of a graph G is a set $S \subseteq V(G)$ such that there exist two connected components L, R of $V(G) \setminus S$ with $N(L) = N(R) = S$.



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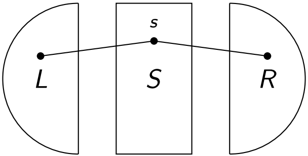


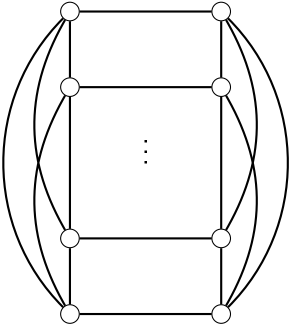
Every $s \in S$ has a neighbor in L and a neighbor in R.

Minimal separators and PMCs

Lemma (Bouchitte and Todinca)
A graph G has polynomially many minimal separators if and only if G has polynomially many potential maximal cliques, and given a list $\mathcal{S}$ of all minimal separators of G, the potential maximal cliques of G can be listed in time polynomial in $|\mathcal{S}|$.

Exponentially many minimal separators

k-prism has $2^k - 2$ minimal separators:



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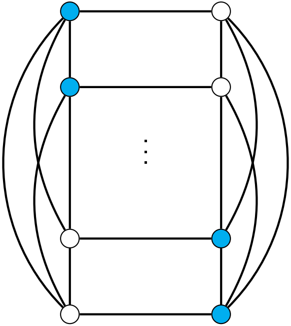

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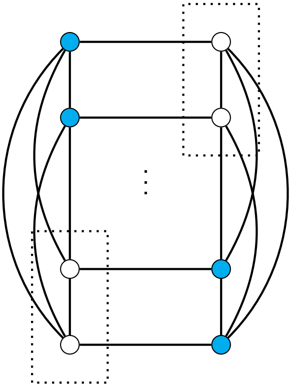
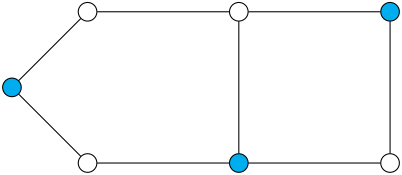


PMC Method Improvement 1

Theorem (Lokshtanov, Vatshelle, Villanger, 2013)
Sufficient to list a subset of PMCs Π such that for every maximal independent set I of G, there exists an **I-good** minimal chordal completion F of G such that every maximal clique of $G + F$ is in Π

I-good Chordal Completions

Let G be a graph and let I be an independent set of G. A minimal chordal completion F is **I-good** if $e \cap I = \emptyset$ for all $e \in F$.



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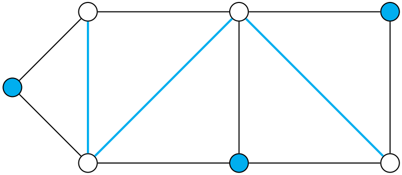

I-good Chordal Completions

Lemma

If Ω is a maximal clique of an I-good minimal chordal completion, then $|\Omega \cap I| \leq 1$.

Lemma

Let G be a graph. For every independent set I of G, there exists an I-good minimal chordal completion of G.

PMC Method Improvement 1

Polynomial-time algorithms were developed for MIS in P_5-free graphs and P_6-free graphs using PMC Method Improvement 1

Can we be even more general than Improvement 1?

PMC Improvement 2: Containers

Let F be an induced subgraph of G. An **F-container** of a set $C \subseteq V(G)$ is a set $A \subseteq V(G)$ such that $C \subseteq A$ and $A \cap F = C \cap F$.

Idea: Find I-containers of minimal separators and potential maximal cliques of G.

PMC Improvement 2: Containers

Theorem (A, Chudnovsky, Pilipczuk, Rzążewski, Seymour)
Sufficient to list a set of subsets $\mathcal{C}$ such that for every maximal independent set I, there exists a minimal chordal completion F of G such that every maximal clique of $G + F$ has an I-container in $\mathcal{C}$

Results

Theorem (A, Chudnovsky, Pilipczuk, Rzążewski, Seymour)
MIS is polynomial-time solvable in long-hole-free graphs.

Theorem (A, Chudnovsky, Pilipczuk, Rzążewski, Seymour)
Feedback Vertex Set is polynomial-time solvable in P_5-free graphs.

MIS Open Problems

Recent progress:

- The methods used to solve MIS in P_5-free and P_6-free graphs don't extent to P_7-free graphs (Grzesik, Klimosova, Pilipczuk, Pilipczuk, 2020)
- Quasi-polynomial time algorithms for MIS in P_t-free graphs and $C_{\geq t}$-free graphs (Garland and Lokshtanov, 2020)

Can containers play a role?

Exponentially many containers

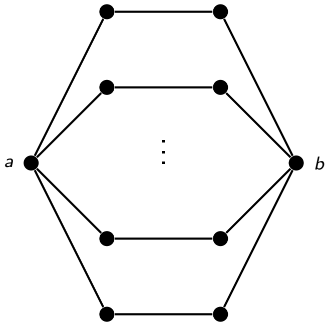

Figure 1: The n-theta is P_7-free but needs exponentially many containers for minimal separators

The End

Questions?