Laplacian Eigenvalues of Cographs

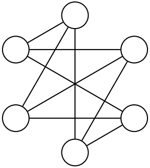
Tara Abrishami
Ed Scheinerman

Joint Mathematics Meetings 2019

Introduction to Cographs

Definition
A graph G is a *cograph* if it can be generated from a single vertex by the operations of disjoint union and join:

- K_1 is a cograph.
- If G and H are cographs, then $G + H$ is a cograph.
- If G and H are cographs, then $G \vee H$ is a cograph.



Cograph Representations

A *cograph representation* of a cograph C tracks the operations of join and disjoint union in the creation of C.

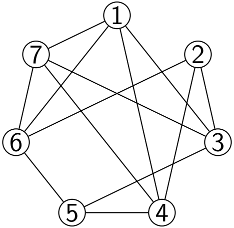


Figure 1: This is a cograph with cograph representation $(6 + 4 + 3) \vee (5 + 2 + 7 \vee 1)$.

Graph Laplacians

Definition
The *Laplacian matrix* of a graph G is given by $L = D - A$, where D is the degree matrix of G and A is the adjacency matrix of G.

- L is positive semidefinite.
- 0 is always an eigenvalue of L.
- The Laplacian spectrum is given by $0 \leq \lambda_2 \leq \ldots \leq \lambda_n$, where $n = |G|$.

Cograph Eigenvalues Are Integers

Proposition
Let C be a cograph. The Laplacian eigenvalues of C are nonnegative integers.

Proposition
Let G be a graph. G is a cograph if and only if for all induced subgraphs H of G, the Laplacian eigenvalues of H are integers.

Central question

What is the combinatorial meaning of cograph eigenvalues?

Threshold Graphs

Definition
A graph G is a *threshold graph* if there exists a weight function of the vertices $f : V(G) \to \mathbb{R}$ such that

$$u \sim v \text{ if and only if } f(u) + f(v) \geq 1.$$

Threshold graphs are a subclass of cographs.

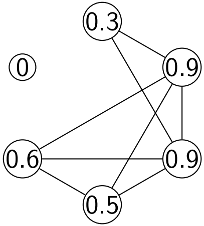


Figure 2: A threshold graph. The vertex labels are the values $f(v)$.

Theorem (Merris)
Let T be a threshold graph. The Laplacian eigenvalues of T are the Ferrer's conjugates of the degree sequence of T.

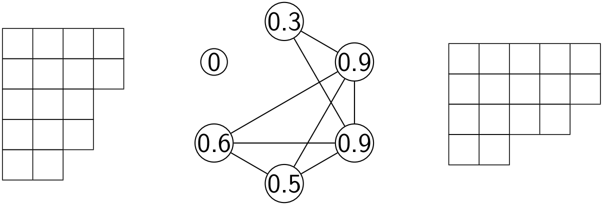


Figure 3:
Degree sequence: [0, 2, 3, 3, 4, 4].
Eigenvalues: $\{0, 0, 2, 4, 5, 5\}$.

Cograph Eigenvalues: Recursive Formula

Proposition (Merris)
Suppose C_1 and C_2 are cographs on n_1 and n_2 vertices, respectively. Let $\Lambda(C_1) = \{0, \lambda_2, \dots, \lambda_{n_1}\}$ and $\Lambda(C_2) = \{0, \gamma_2, \dots, \gamma_{n_2}\}$.

The spectrum of $C_1 + C_2$ is

$$\Lambda(C_1 + C_2) = \{0, 0, \lambda_2, \dots, \lambda_{n_1}, \gamma_2, \dots, \gamma_{n_2}\}.$$

The spectrum of $C_1 \vee C_2$ is

$$\Lambda(C_1 \vee C_2) = \{0,\ \lambda_2 + n_2, \dots, \lambda_{n_1} + n_2, \\ \gamma_2 + n_1, \dots, \gamma_{n_2} + n_1, n_1 + n_2\}.$$

Cograph Eigenvalue Results

Proposition (Citation)
For any graph G, $\lambda_2(G) \leq \kappa(G)$, where $\lambda_2(G)$ is the second-smallest Laplacian eigenvalue of G, and $\kappa(G)$ is the vertex connectivity of G.

Proposition
Let C be a cograph. Then, $\lambda_2(C) = \kappa(C)$.

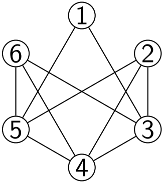
Cograph Eigenvalue Results

Proposition

Let C be a cograph, and let $\lambda_2(C)$ be the second-smallest eigenvalue of C. Then, $\lambda_2(C)$ is unique if and only if C has a unique minimum cut set S and $C - S$ has exactly two components.

If $\lambda_2(C)$ is unique, then the Fiedler vector of C, $v_2(C)$, defines a partition of $V(C)$ into three sets: the unique minimum cut set S and the two components of $C - S$.

Fiedler Vector Example



$$v_2(C) = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Figure 4: Example cograph C and its Fiedler vector $v_2(C)$. $\Lambda(C) = \{0, 2, 3, 4, 5, 6\}$.

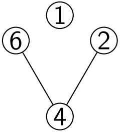


Figure 5: $C - S$, where $S = \{3, 5\}$.

Definition
Let G be a graph. Two vertices v and w are *twins* if $N(v) - w = N(w) - v$.

Proposition
The property of being twins is an equivalence relation on the vertex set of G.

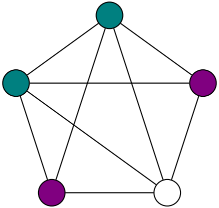


Figure 6: Example of twin vertices in a cograph.

Cograph Eigenvalue Results

Definition
The *twin partition* of a graph G is the unique partition of the vertex set such that every pair of vertices in the same part are twin vertices. The *twin numbers* of a graph G are the sizes of the parts of the twin partition.

Remark
Every twin part is either a clique or an independent set.

Every pair of vertices in the same twin part have the same degree.

Cograph Eigenvalue Results

Theorem
Let C be a cograph. Every Laplacian eigenvalue of C is the sum of twin numbers of C. Further, let $\{P_1, \ldots, P_n\}$ be the twin partition of C, and let $\{|P_1|, \ldots, |P_n|\}$ be the twin numbers of C. Every twin number $|P_k|$ contributes to d eigenvalues, where $d = d(v)$ for $v \in P_k$ is the degree of the vertices in P_k.

Cograph Eigenvalue Example

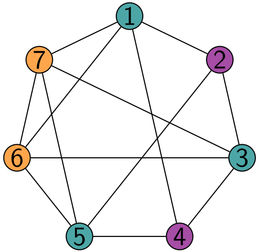


Figure 7: Cograph given by $(\mathbf{1} + \mathbf{3} + \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$. Its spectrum is $\{0, 3, 3, 4, 4, 5, 7\}$.

	2	**2**	**3**
0			
3			x
3			x
4	x	x	
4	x	x	
5		x	x
7	x	x	x

Table 1: Chart showing which twin number contributes to which eigenvalue.

Threshold Eigenvalue Example

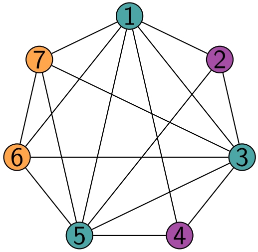


Figure 8: Threshold graph given by $(\mathbf{1} \vee \mathbf{3} \vee \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$. Its spectrum is $\{0, 3, 3, 5, 7, 7, 7\}$.

	2	2	3
0			
3			x
3			x
5		x	x
7	x	x	x
7	x	x	x
7	x	x	x

Table 2: Chart showing which twin number contributes to which eigenvalue.

Definition
The *twin reduction graph* of a cograph C is the induced subgraph formed by taking one vertex from each twin class of C.

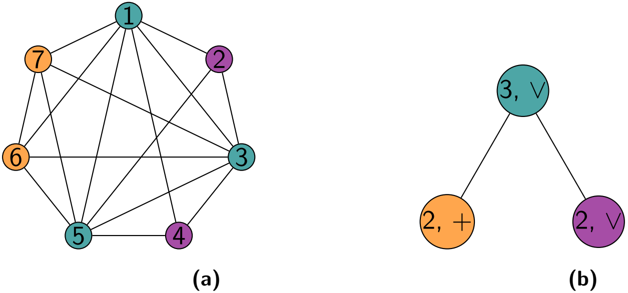


Figure 9: (a): A cograph given by $(\mathbf{1} \vee \mathbf{3} \vee \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$. (b) The twin reduction graph. The vertex labels are the twin numbers and the operations of the twin classes.

Twin Reduction Conjecture

Definition
Let $R(C_1)$ and $R(C_2)$ be twin reduction graphs of cographs C_1 and C_2. We say $R(C_1)$ and $R(C_2)$ are *equivalent* if the graphs are isomorphic and the twin numbers of the vertices are identical.

Proposition
Let C_1 and C_2 be cographs with equivalent reduction graphs $R(C_1)$ and $R(C_2)$. Let k be the number of twin numbers of C_1 and C_2. Let I be the set of twin numbers whose operations are identical in $R(C_1)$ and $R(C_2)$. Then, C_1 and C_2 have at least

$$k + \sum_{i \in I} (|P_i| - 1)$$

eigenvalues in common.

Twin Reduction Example

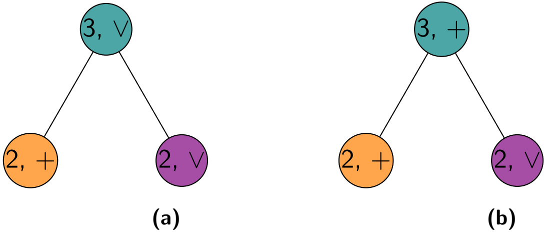


Figure 10: (a): Twin reduction graph of a cograph given by $(\mathbf{1} \vee \mathbf{3} \vee \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$ with spectrum $\{0, 3, 3, 5, 7, 7, 7\}$. (b): Twin reduction graph of a cograph given by $(\mathbf{1} + \mathbf{3} + \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$ with spectrum $\{0, 3, 3, 4, 4, 5, 7\}$.

Acknowledgements

Thank you to Dr. Scheinerman!