Even-hole-free graphs with bounded degree have bounded treewidth

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Introduction

G is **even-hole-free** if G does not have an induced cycle of even length.

Conjecture (Aboulker, Adler, Kim, Sintiari, Trotignon).
Even-hole-free graphs with bounded degree have bounded treewidth.

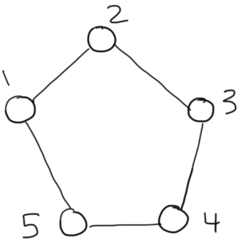
Tree decompositions

A **tree decomposition** (T, χ) of a graph G is a tree T and a map $\chi : V(T) \to 2^{V(G)}$, such that

1. for all $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$
2. for all $v_1 v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$
3. for all $v \in V(G)$, the set $\{t \in V(T) : v \in \chi(t)\}$ induces a connected subtree of T

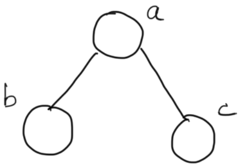
The **width** of (T, χ) is $\max_{t \in V(T)} |\chi(t)| - 1$.

The **treewidth** of G is the minimum width of a tree decomposition of G.



Tree decomposition (T, χ):

T:



$$\chi(a) = \{2, 5, 4\}$$

$$\chi(b) = \{2, 3, 4\}$$

$$\chi(c) = \{2, 1, 5\}$$

<u>width</u> of $(T, \chi) = 2$

$(k, S, c)^*$-separators

A set $X \subseteq V(G)$ is a **$(k, S, c)^*$-separator** if

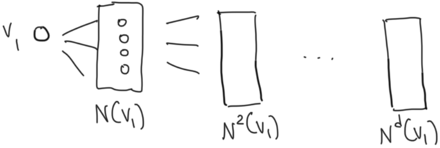
- $|X| \leq k$
- every component of $G \setminus X$ has at most $c|S|$ vertices of S

$\text{sep}^*_c(G) :=$ min k such that G has a $(k, S, c)^*$-separator for every $S \subseteq V(G)$

Theorem (Harvey, Wood)
For all $c \in [\frac{1}{2}, 1)$,

$$sep^*_c(G) \leq tw(G) + 1 \leq \frac{1}{1-c} sep^*_c(G).$$

A set Y is **d-bounded** if there exists $v_1, \ldots, v_d \in V(G)$ such that $Y \subseteq N^d[v_1] \cup \ldots \cup N^d[v_d]$



$= 1 + \delta + \delta^2 + \ldots + \delta^d.$

So, $|Y| \leq d + d\delta + dd^2 + \ldots + d\delta^d$

A set $Y \subseteq V(G)$ is a **(w, c, d)-balanced separator** if

- Y is d-bounded
- $w(Z) \leq c$ for every component Z of $G \setminus Y$

max possible size of d-bounded set

Lemma
Let δ, d be positive integers, let $c \in [\frac{1}{2}, 1)$, let $\Delta = d + \delta d + \ldots + \delta^d d$. Let G be a graph with maximum degree δ. Suppose that for all $w : V(G) \to [0, 1]$ such that $w(G) = 1$ and $w^{\max} < \frac{1}{\Delta}$, G has a (w, c, d)-balanced separator. Then, $tw(G) \leq \frac{1}{1-c}\Delta$.

Proof.

Want to show that G has a $(\Delta, S, c)^*$-separator for every $S \subseteq V(G)$.

- If $|S| \leq \Delta$, then S is a $(\Delta, S, c)^*$-separator of G.
- If $|S| > \Delta$, let $w_S : V(G) \to [0, 1]$ be such that

$$w_S(v) = \begin{cases} \frac{1}{|S|} & v \in S \\ 0 & v \notin S \end{cases}$$

Proof (continued)

Then, $w_S^{max} < \frac{1}{\Delta}$, so G has a (w_S, c, d)-balanced separator Y.

- Let Z be a component of $G \setminus Y$. Since $w_S(Z) \leq c$, it follows that Z has at most $c|S|$ vertices of S
- $|Y| \leq \Delta$

So, Y is a $(\Delta, S, c)^*$-separator of G.

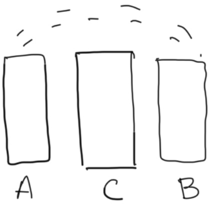
□

To prove bounded treewidth, we focus on finding (w, c, d)-balanced separators.

Separations

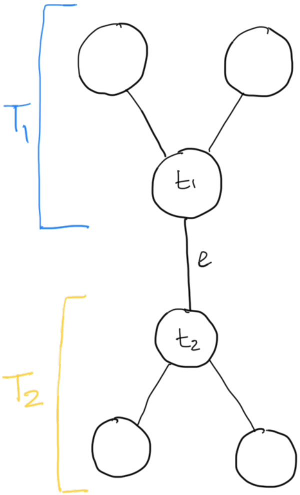
A **separation** of G is a triple (A, C, B) such that

- A, C, B are disjoint
- $A \cup C \cup B = V(G)$
- A is anticomplete to B



Every edge of a tree decomposition (T, χ) of G corresponds to a separation (A, C, B) of G.

Separations and tree decompositions



Separation (A, C, B):

$$C = \chi(t_1) \cap \chi(t_2)$$

$$A = \bigcup_{t \in T_1} \chi(t) \setminus C$$

$$B = \bigcup_{t \in T_2} \chi(t) \setminus C$$

Pf: Suppose $u \in A$, $\in B$, $uv \in E(G)$.

- $\exists t$ s.t. $u, v \in \chi(t)$
- if $t \in T_1$, $v \in \chi(t_1) \cap \chi(t_2)$
- if $t \in T_2$, $u \in \chi(t_1) \cap \chi(t_2)$

Laminar collections

Two separations $S_1 = (A_1, C_1, B_1)$ and $S_2 = (A_2, C_2, B_2)$ are **non-crossing** if, up to symmetry,

- $A_1 \cup C_1 \subseteq A_2 \cup C_2$
- $B_2 \cup C_2 \subseteq B_1 \cup C_1$

A collection $\mathcal{S}$ of separations is **laminar** if every pair of separations in $\mathcal{S}$ is non-crossing.

Theorem (Robertson, Seymour)
If $\mathcal{S}$ is a collection of laminar separations of G, then there exists a tree decomposition (T, χ) such that there is a one-to-one correspondence between edges of T and separations in $\mathcal{S}$

If $C \subseteq N[K]$, then $C \subseteq N^2[v]\ \forall v \in K$

↓

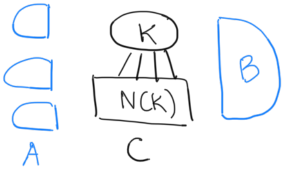
Star separation: (A, C, B) such that $C \subseteq N[K]$ for some clique K

Let $c \in [\frac{1}{2}, 1)$. We may assume that if (A, C, B) is a star separation of G, then:

- B is connected
- $w(A) < 1 - c$

B: largest connected component of $G \setminus N[K]$

$\Rightarrow w(B) > c$



Central bag

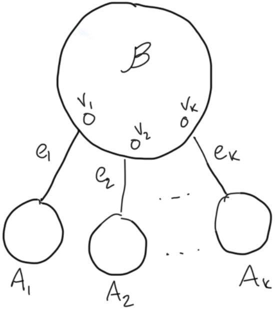
Suppose $\mathcal{S}$ is a laminar collection of separations and $(T_{\mathcal{S}}, \chi_{\mathcal{S}})$ is the tree decomposition corresponding to $\mathcal{S}$. Then, there is a bag $\beta = \chi(t_0)$ such that:

(1) If $(A, C, B) \in \mathcal{S}$, then $\beta \cap A = \emptyset$

(2) $G[\beta]$ does not have a (w', c, d')-balanced separator

We call β the **central bag**

Central bag



1) $e_i \Rightarrow (A_i, C_i, B_i)$, with $C_i \subseteq N^2[v_i]$

2) $N(A_i) \cap \mathcal{B} \subseteq C_i$

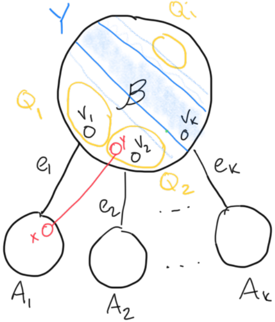
$w' : V(B) \to [0,1]$, s.t.

$$w'(v) = \begin{cases} w(v_i) + w(A_i) & \text{if } v \in \{v_1, \dots, v_k\} \\ w(v) & \text{otherwise} \end{cases}$$

- $w'(B) = w(G) = 1$

Claim: If $\mathcal{B}$ has a $(w', c, d-2)$-balanced separator Y, then $N^2[Y]$ is a (w, c, d)-balanced separator of G

Central bag



Components of $G \setminus N^2[Y]$:

- $Q_i \cup (\bigcup_{v_j \in Q_i} A_j) \setminus N^2[Y] = Z_i$.
- A_i s.t. $v_i \in Y$

1) Z_i is anticomplete to Z_j.
   Edge xy means $y \in C_1$, so $y \in N^2[v_1]$
   There is a path $v_1 - u - y$, with $u \in Y$.
   But then $y \in N^2[Y]$.

2) If $v_i \in Y$, then $N(A_i) \subseteq N^2[Y]$.
   $N(A_i) \subseteq C_i$,
   $C_i \subseteq N^2[v_i]$
   $N^2[v_i] \subseteq N^2[Y$

Proof outline

The central bag β for $\mathcal{S}$ has two important properties:

- β does not have a (w', c, d')-balanced separator
- β is "simpler" than G

↑ because $\beta \cap A = \emptyset$ for all $(A, C, B) \in \mathcal{S}$.

Forcers

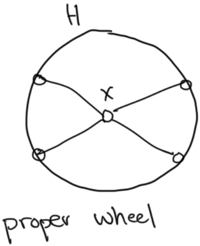
Which separations are important for even-hole-free graphs?

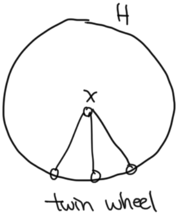
- star cutsets: $C \subseteq N[v]$ for some $v \in V(G)$
- double star cutsets: $C \subseteq N[u] \cup N[v]$ for some $uv \in E(G)$

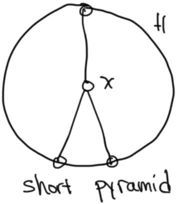
A **forcer** $F = (H, K)$ is a hole H and a clique K of size one or two such that $N[K]$ is a star cutset or a double star cutset of G.

There are three types of forcers in even-hole-free graphs:

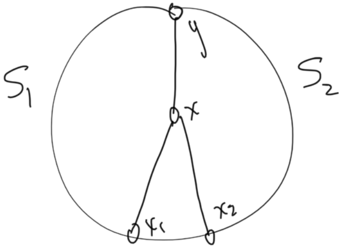
1. $F = (H, \{x\})$ where (H, x) is a proper wheel
2. $F = (H, \{x\})$ where (H, x) is a twin wheel
3. $F = (H, \{x, y\})$, where (H, x) is a short pyramid







Forcer example



short pyramid

<u>Lemma</u>: $N(x) \cup N(y)$ is a cutset of G that separates S_1 from S_2.

<u>Note</u>: F touches two connected comps of $G \setminus (N(x) \cup N(y))$.

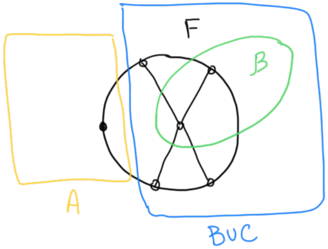
Let (A, C, B) be the separation induced by F, so $C \subseteq N(x) \cup N(y)$.

B is connected $\Rightarrow F \cap A \neq \emptyset$.

Lemma
Let F be a forcer and let (A_F, C_F, B_F) be the star separation induced by F. Then, $F \cap A_F \neq \emptyset$.

Recall that if β is the central bag for a laminar collection of separations $\mathcal{S}$, then $\beta \cap A = \emptyset$ for all $(A, C, B) \in \mathcal{S}$. Therefore:

- If $(A_F, C_F, B_F) \in \mathcal{S}$ and β is the central bag for $\mathcal{S}$, then β does not contain F.



Forcers and bounded treewidth

Theorem
Let G be an even-hole-free graph with maximum degree δ and no forcers. Then, G does not have a star cutset.

Theorem
Let G be an even-hole-free graph with maximum degree δ and no star cutset. Then, G has a (w, c, d)-balanced separator.

$\Longrightarrow$ If G has no forcers, then G has a (w,c,d)-balanced separator

Proof Sketch

Proof sketch:

1. Let $\mathcal{F}$ be all forcers of G and let $\mathcal{S}$ be the set of separations induced by forcers of G.
2. Use $\mathcal{S}$ to find an induced subgraph β of G such that β does not contain any forcer in $\mathcal{F}$.
3. Then, β has a (w', c, d')-balanced separator
4. Therefore, G has a (w, c, d)-balanced separator

↳ What if S is not laminar?

Separation dimension

Lemma
Let $S_1 = (A_1, C_1, B_1)$ *and* $S_2 = (A_2, C_2, B_2)$ *be two star separations with* $C_1 \subseteq N[K_1]$ *and* $C_2 \subseteq N[K_2]$*. If* K_1 *is anticomplete to* K_2*, then* S_1 *and* S_2 *are non-crossing.*

Because G has bounded maximum degree, we can partition $\mathcal{S}$ into a bounded number of laminar collections.

Separation dimension of $\mathcal{S}$: the min k such that $\mathcal{S}$ can be partitioned into k laminar collections

Proof Sketch

Proof sketch:

1. Let $\mathcal{F}$ be the set of all forcers in G.
2. Can partition $\mathcal{F}$ into $k = f(\delta)$ sets $\mathcal{F}_1, \ldots, \mathcal{F}_k$, such that the collection of separations $\mathcal{S}_i$ is laminar.
3. Find β_1, the central bag for $\mathcal{S}_1$. Then:
   - β_1 does not have any forcers in $\mathcal{F}_1$
   - β_1 does not have a (w_1, c, d_1)-balanced separator

Proof Sketch

4. Iteratively find β_i, the central bag for $\mathcal{S}_i$ restricted to β_{i-1}. Then:
   - β_i does not have any forcers in $\mathcal{F}_1, \ldots, \mathcal{F}_i$
   - β_i does not have a (w_i, c, d_i)-balanced separator
5. Finally, β_k does not have any forcers, and β_k does not have a (w_k, c, d_k)-balanced separator.
6. Because β_k does not have any forcers, β_k has a (w_k, c, d_k)-balanced separator.

□

Key Ideas

Key ideas:

1. Can find a **central bag** β such that β has lower dimension than G and β has a (w', c, d')-balanced separator only if G has a (w, c, d)-balanced separator
2. Can find **forcers** that induce bounded separations in G
3. Bounded degree means separations can be partitioned into a bounded number of laminar collections
4. Graphs with no forcers are "simple," so we can prove properties of graphs with no forcers

Thank you!

Questions?