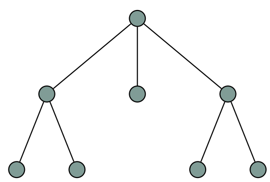
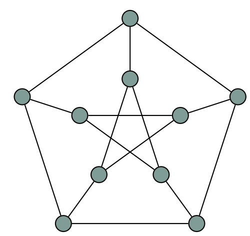
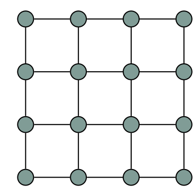
Math 107: Introduction to Graph Theory

Contents

Introduction

Welcome! These are the course notes for Math 107: Introduction to Graph Theory, taught by me (Tara Abrishami) at Stanford University in Winter 2026.

The purpose of these notes falls into three categories:

- To list and define the words and concepts you need to know for each class.
- To list the content we will cover in each class.
- To direct you toward additional resources you can consult.

The additional resources are drawn from the 6th edition of Reinhard Diestel's book *Graph Theory*, which is available for free download through the Stanford University library, and from the recorded video lectures of his class, which closely follow the book, available on Youtube. The definitions in these notes and the content of this course also follow the book, so together these should form a comprehensive and consistent set of resources.

Reading the course notes before each class is a requirement. All of the other materials are optional. If you enjoy reading textbooks, you can consult the book; if you learn well from watching videos, you can watch the video lectures (and Diestel is an engaging lecturer). If you learn best by doing, you can focus on thinking through the extra exercises given each week. Please come to office hours with questions about any of this!

These notes will be continuously updated as we go along, so please make sure to refresh the page!

Last update: February 20, 2026.

1. Graphs, paths, cycles

1.1. January 6th.

- Graph, vertex, edge
- Adjacency, incidence, degree, neighborhood
- Minimum degree, maximum degree, average degree
- Subgraph, induced subgraph, spanning subgraph
- Graph isomorphism
- Complement of a graph

A (simple) **graph** $G = (V, E)$ consists of a set of **vertices** V and a set of **edges** E, where each edge is a pair of distinct vertices, called the **ends** of the edge. When G is a graph, we use $V(G)$ and $E(G)$ to refer to its vertex set and edge set, respectively. Two vertices v_1 and v_2 of a graph G are **adjacent** if there is an edge of G with ends v_1 and v_2. We denote this edge as v_1v_2. A vertex v of G is **incident** to an edge e of G if v is an end of e. Likewise, an edge e is **incident** to a vertex v of G if v is an end of e.

The **degree** of a vertex v of G, denoted $d(v)$, is the number of edges that v is incident to in G. The **neighborhood** of a vertex v, denoted $N(v)$, is the set of vertices u of G such that v is adjacent to u, i.e. uv is an edge of G. The size of the neighborhood of a vertex is equal to its degree. A vertex is **isolated** if it has degree zero.

The **minimum degree** of a graph G, denoted $\delta(G)$, is the minimum degree $d(v)$ of a vertex v of G. Similarly, the **maximum degree** of G, denoted $\Delta(G)$, is the maximum degree $d(v)$ of a vertex v of G. The **average degree** of G, denoted $d(G)$, is the average degree over all of its vertices; so $d(G) \coloneqq \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v)$.

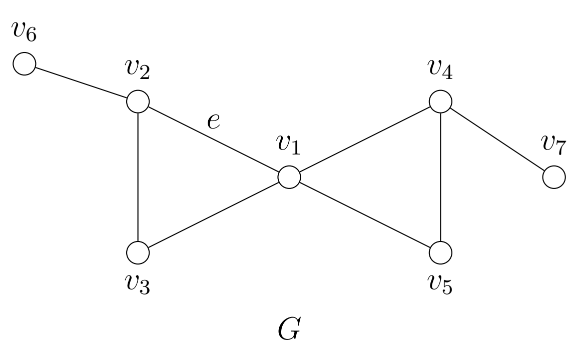


Figure 1. In this example graph G, vertex v_1 is adjacent to vertices v_2, v_3, v_4, and v_5. Vertices v_1 and v_2 are both incident with edge e, and v_1 and v_2 are also the ends of e. The neighborhood of v_1 is $N(v_1) = \{v_2, v_3, v_4, v_5\}$, and its degree is $d(v_1) = 4$.

A graph H is a **subgraph** of a graph G if H is formed from G by deleting vertices and edges. A graph H is an **induced subgraph** of a graph G if H is formed from G by deleting vertices. Observe that every induced subgraph is a subgraph, but not every subgraph is an induced subgraph. A subgraph H of a graph G is **spanning** if the vertex set of H equals the vertex set of G.

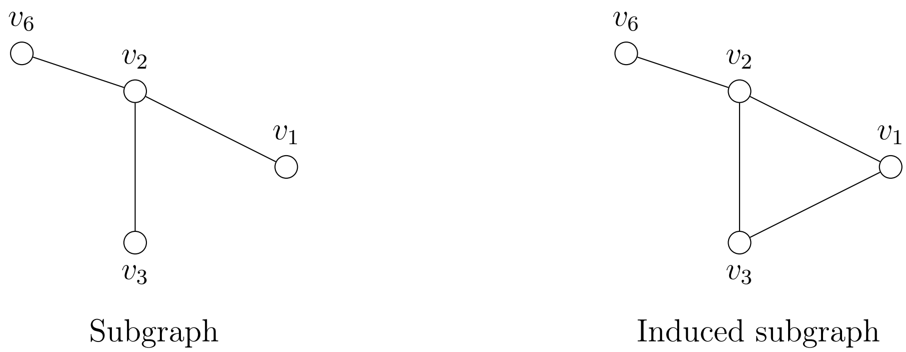


FIGURE 2. This figure shows a subgraph of G on the left and an induced subgraph of G on the right. Neither of these subgraphs is spanning.

Let H and H' be graphs. An **isomorphism** from H to H' is a bijection $\varphi : H \to H'$ such that uv is an edge of H if and only if $\varphi(u)\varphi(v)$ is an edge of H'. Two graphs H and H' are **isomorphic** if there is an isomorphism from H to H'. We typically only care about graphs up to isomorphism. For example, since the graph H' in Fig. 3 is isomorphic to the graph H and H is a subgraph of G in Fig. 1, we would consider G to contain H' as a subgraph.

FIGURE 3. The graphs H and H' are isomorphic. Every isomorphism from H to H' sends v_2 to b. Any bijective mapping from $\{v_6, v_3, v_1\}$ to $\{a, c, d\}$ will create an isomorphism. For example, the mapping $\varphi : H \to H'$ such that $\varphi(v_6) = a, \varphi(v_3) = d, \varphi(v_1) = c, \varphi(v_2) = b$ is an isomorphism.

The **complement** of a graph $G = (V, E)$, denoted $\overline{G}$, is the graph with vertex set V and such that distinct vertices u and v in V are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.

1.1.1. **Class plan.** In class we will go over the following problems.

1. Write a formula for the average degree $d(G)$ of a graph $G = (V, E)$ in terms of $|V|$ and $|E|$.
2. For a graph $G = (V, E)$, write a formula for the number of edges of the complement $\overline{G}$ in terms of $|V|$ and $|E|$.
3. Is there a 3-regular graph with 9 vertices?
4. Prove that every graph with at least two vertices contains two distinct vertices with the same degree.

1.1.2. **References.** Optional references for today's class:

- Diestel Sections 1.1, 1.2
- Diestel Graph Theory:
  - Lecture 1: Introduction
  - Lecture 2: Invariants I up to minute 38:00.

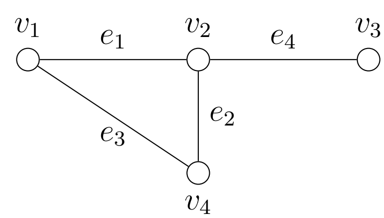
1.1.3. **Class log.**

- We discussed the problems above
- We also discussed the graph reconstruction problem. We observed that the number of vertices, number of edges, and degrees of each vertex can be reconstructed, as long as the graph has at least three vertices. We also observed that reconstruction fails for graphs with two vertices: the two graphs with two vertices are indistinguishable.

1.2. **January 8th.**

- Walk, trail, path
- Connected, disconnected
- Distance, diameter
- Cycle, induced cycle, girth
- Acyclic, forest, tree, leaf
- Cut vertex, cut edge
- k-connected

Let G be a graph. A **walk** in G is a sequence of alternating vertices and edges $v_1e_1v_2\dots v_{k-1}e_{k-1}v_k$, such that each edge e_i has ends v_i and v_{i+1}. The vertices v_1 and v_k are the **ends** of the walk. A walk is **closed** if its two ends are the same. A **trail** of G is a walk where no edge is used more than once.



G

$$W = v_1e_1v_2e_2v_4e_3v_1e_1v_2e_4v_3$$

FIGURE 4. An example walk W in a graph G with ends v_1 and v_3.

A **path** is a graph $P = (V, E)$ with vertex and edge set

$$V = \{v_1, v_2, \dots, v_k\}, \quad E = \{v_1v_2, v_2v_3, v_3v_4, \dots, v_{k-1}v_k\}.$$

The vertices v_1 and v_k are the **ends** of the path P. The **length** of a path P is the number of its edges. We will often denote a path with k vertices as P_k. (Observe that the path P_k has length $k-1$, not k.) Notice that every path defines a walk.

A path in a graph G is a subgraph of G isomorphic to a path. A walk in G defines a path if and only if no vertex is used more than once.

A graph G is **connected** if G contains a path with ends u and v for every pair of distinct vertices u and v in G. Intuitively, G is connected if you can move from any vertex to any other vertex by traversing the edges of G. A graph G is **disconnected** otherwise. A **connected component** of a graph G is a maximal connected subgraph of G.

If vertices u and v are the ends of a path P or a walk W, we say that P is a path from u to v and W is a walk from u to v. We may also say that P is a u-v path or W is

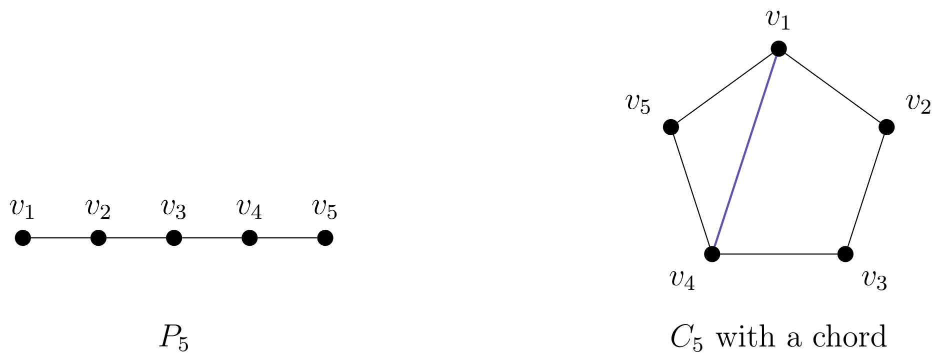


FIGURE 5. This figure shows the path of length 5, P_5, on the left, and the cycle of length 5, C_5, on the right. The purple edge is not part of the cycle, it is a chord.

u-v walk. The **distance** between two vertices u and v of a graph G, denoted $d(u, v)$, is the length of the shortest path from u to v in G. The **diameter** of a graph G, denoted $\mathrm{diam}(G)$, is the maximum distance between any two vertices of G.

A **cycle** is a graph $C = (V, E)$ with vertex and edge set

$$V = \{v_1, v_2, \ldots, v_k\}, \quad E = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{k-1}v_k, v_kv_1\},$$

with $k \geq 3$. Deleting a single edge from a cycle yields a path. The **length** of a cycle is the number of its edges. We will often denote a cycle with k vertices as C_k. (Observe that the cycle C_k has length k.) Notice that every cycle defines a closed walk. The cycle C_3 of length three is also called a **triangle**.

A cycle in a graph G is a subgraph of G isomorphic to a cycle. The **girth** of a graph G, denoted $g(G)$, is the minimum length of a cycle in G. A **chord** of a cycle in a graph G is an edge of G between two vertices v_i and v_j such that v_iv_j is not an edge of the cycle. A cycle in a graph G that has no chords is an **induced cycle**.

A graph with no cycles is called **acyclic**; an acyclic graph is also called a **forest**. A graph is a **tree** if it is connected and acyclic. A vertex of degree one in a tree is called a **leaf**.

Let G be a graph. A **cut vertex** of G is a vertex v such that deleting v disconnects the connected component of G containing v. A **cut edge** of G is an edge $e = uv$ such that u and v are in different connected components of G_e, the graph formed by deleting e.

For an integer $k \geq 2$, a graph G is **k-connected** if $|V(G)| > k$ and for every set X of vertices of G of size less than k, the graph $G - X$ formed by deleting the vertices in X is connected. A graph is k-connected if you must delete at least k vertices to disconnect the graph. The **connectivity** of a graph G, denoted $\kappa(G)$, is the maximum integer k for which G is k-connected.

1.2.1. **Class plan.** In class we will go over the following problems.

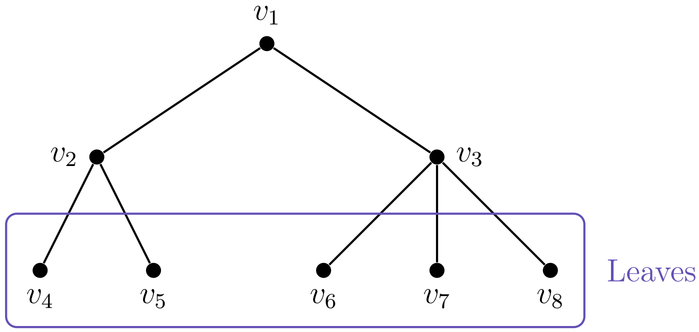


FIGURE 6. An example tree with eight vertices and five leaves.

1. Show that the components of a graph partition its vertex set.

2. Connectivity true or false:
   - Is every graph with minimum degree two 2-connected?
   - Does every 2-connected graph contain a cycle?
   - Is every graph that contains a cycle 2-connected?
   - Does every graph with minimum degree two contain a cycle?
   - Is $\kappa(G) \leq \delta(G)$ for every graph G?

3. Prove that every tree with at least two vertices has more leaves than vertices of degree at least three.

1.2.2. **References.** Optional references for today's class:

- Diestel Sections 1.3, 1.4, 1.8
- Diestel Graph Theory:
  - Lecture 2 from minute 38:00 to 1:04:00.
  - For more advanced material about connectivity: Lecture 2 from minute 1:04:00 to the end, and Lecture 3

1.2.3. **Class log.**

- We discussed the problems above
- We corrected the definition of k-connected. It is now correct in the notes above. To reiterate: a graph G is **k-connected** if $|V(G)| > k$ and if $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.
- I've included two notes on problems from class below.
- I also included some notes on the graph reconstruction conjecture, since some people asked. **These are not necessary for class, they are just for fun if you are interested!**
- Also, here is a recent Numberphile video on graph reconstruction. This is also just for fun!

Notes from class.

First, let's redo the proof of the last connectivity true/false question using the correct definition of k-connected.

Claim 1.1. For every graph G,

$$\kappa(G) \leq \delta(G).$$

Proof. Let G be a graph and let v be a vertex of G of minimum degree. By the definition of k-connected, $\kappa(G) \leq |V(G)| - 1$. Let $X = N(v)$ be the set of neighbors of v, so $|X| = \delta(G)$. If $|V(G)| = \delta(G) + 1$, i.e. if the whole graph consists of v and its neighbors, then $\kappa(G) \leq |V(G)| - 1 = \delta(G)$, as desired.

Therefore, we may assume that $|V(G)| > \delta(G) + 1$, so G contains some vertex w that is non-adjacent to v. In particular, w is not in X. Now, the graph $G - X$ formed by deleting X yields a graph with at least two connected components, one consisting of the vertex v and one containing the vertex w. This proves that $\kappa(G) \leq \delta(G)$. □

I've next written down a proof of the last statement from class. I have added the assumption that we are working with trees with at least two vertices to better illustrate the base case.

Claim 1.2. Every tree with at least two vertices has more leaves than vertices of degree at least three.

Proof. We prove the claim by induction on the number of vertices. The base case is when the tree has two vertices. There is a single two-vertex tree, the graph consisting of two vertices with an edge between them. This tree has two leaves and zero vertices of degree at least three, so the statement holds for the base case.

Let us assume that the statement holds for all trees with n vertices, for $n \geq 2$. We will show using this assumption that the statement holds for trees with $n + 1$ vertices.

Consider a tree T with $n+1$ vertices. Let v be a leaf of T, and let T' be the tree formed by deleting v from T. Now, T' is a tree with n vertices, so, by the induction hypothesis, T' has more leaves than vertices of degree three.

What happens to the number of leaves and the number of vertices of degree at least three when we move from T' to T? Let u be the (unique) neighbor of v in T. Every vertex of T' except u has the same degree in T as in T'. Therefore, T has one more leaf than T', unless u is a leaf of T', in which case T has the same number of leaves as T'. Also, T has the same number of vertices of degree at least three as T', unless u has degree two in T', in which case T has one more vertex of degree at least three as T'. In particular, if T has one more vertex of degree at least three than T', then T also has one more leaf than T'. Therefore, T has more leaves than vertices of degree at least three. This completes the proof. □

If you want to see the ending in more detail, you can also do out the cases. For example, we could replace the last paragraph above with the following:

Cases for u. Let u be the unique neighbor of v in T. There are three possibilities: u is a leaf of T', u has degree two in T', or u has degree at least three in T'.

First, suppose that u is a leaf in T'. Then, T has the same number of leaves as T': the vertex v is a leaf of T but not of T', and the vertex u is a leaf of T' but not of T. The number of vertices of degree at least three is also the same in T and in T'. Therefore, the statement holds for T in this case.

Next, suppose that u has degree two in T'. Then, T has one more leaf than T': every leaf of T' is also a leaf of T, and v is a leaf of T and not of T'. Also, T has one more vertex of degree at least three than T': every vertex of degree at least three in T' has degree at least three in T, and u has degree three in T but not in T'. Therefore, the statement holds for T in this case.

Finally, suppose that u has degree at least three in T'. Then, T has the same number of vertices of degree at least three as T'. Also, T has one more leaf than T': every leaf of T' is a leaf of T, and v is a leaf of T but not of T'. Therefore, the statement holds for T in this case.

Since the statement holds in all cases, this proves that the statement holds for T. This completes the proof by induction. □

Graph Reconstruction notes.

This section is just for fun! You do not need to know this for class.

First let's review the set-up of graph reconstruction. Let G be a graph. The set of vertex-deleted subgraphs of G is called the **deck** of G. Specifically, the deck of G is the set of graphs $\{G - v \mid v \in V(G)\}$, where $G - v$ is the graph formed by deleting v. A graph G is **reconstructible** if G is uniquely determined (up to isomorphism) by its deck.

Here are some types of graphs which are known to be reconstructible.

- Graphs where every vertex has the same degree (these are called **regular** graphs)
- Graphs that contain a vertex adjacent to every other vertex
- Disconnected graphs
- Trees

The first two have simple proofs that you can try to find on your own. The second two are harder. Come to office hours if you have questions about any of this or want to see the proofs!

2. Trees and other basic notions

2.1. January 13th.

- Spanning tree
- Bipartite graph, r-partite graph

A **spanning tree** of a graph G is a spanning subgraph T of G such that T is a tree.

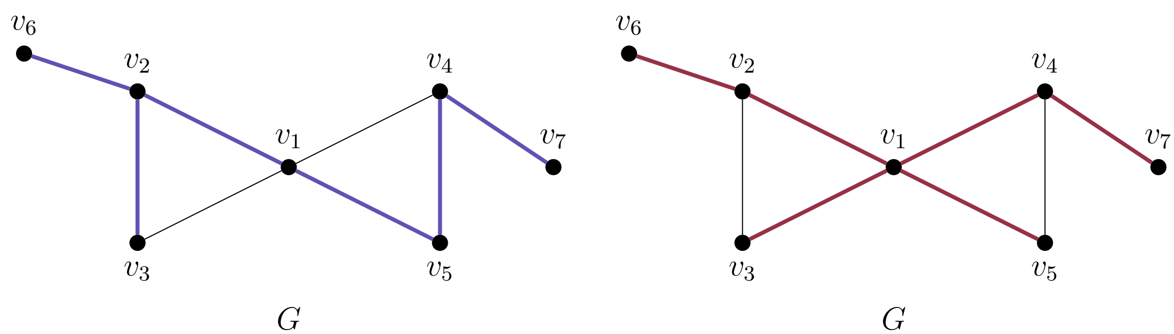


Figure 7. Two different spanning trees, one in purple on the left and one in red on the right, of a graph G.

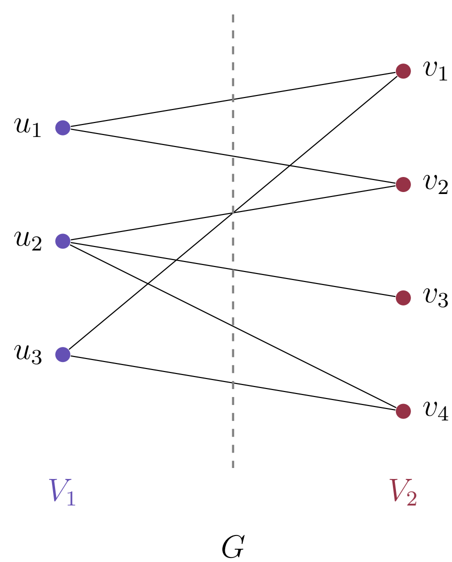


Figure 8. An example of a bipartite graph G with bipartition V_1, V_2.

A graph $G = (V, E)$ is **bipartite** if the vertex set of G can be partitioned into two parts V_1, V_2, called a **bipartition** of V, such that every edge of G has one end in V_1 and one end in V_2. Given an integer $r \geq 2$, a graph $G = (V, E)$ is **r-partite** if the vertex set of G

can be partitioned into r parts $V_1, \dots, V_r$ such that every edge of G has ends in distinct parts.

2.1.1. **Class plan.** In class we will go over the following problems.

1. Prove that every connected graph has a spanning tree.
2. Prove that a graph G is bipartite if and only if every cycle of G is even.
3. Prove the following characterizations of trees.

   Theorem 2.1. The following are equivalent for every graph T.
   (a) T is a tree.
   (b) For every two vertices u, v of T, there is a unique path of T from u to v.
   (c) T is connected but for every edge e of T the graph $T - e$ formed by deleting edge e is disconnected.
   (d) T has no cycle but for every pair of non-adjacent vertices u, v of T, the graph $T + xy$ formed by adding edge xy to T contains a cycle.
   (e) T is connected and $|E(T)| = |V(T)| - 1$.
   (f) T is acyclic and $|E(T)| = |V(T)| - 1$.

2.1.2. **References.** Optional references for today's class:

- Diestel Sections 1.5, 1.6
- Diestel Graph Theory:
  - Lecture 4
  - Lecture 5 up to minute 30:00

2.1.3. **Class log.**

- We proved question 1 above using a constructive proof:

  Claim 2.2. Every connected graph has a spanning tree.

  Proof. Let G be a connected graph and let v be a vertex of G. Let d be the maximum distance of a vertex of G from v. Let L_i denote the set of vertices of G of distance i from v, for i from 0 to d. (Observe that $L_0 = \{v\}$.) Let H be the graph with vertex set $V(G)$ and edge set defined as follows. For every $i \in 1, \dots, d$ and for every vertex u in L_i, choose one neighbor u' of u with $u' \in L_{i-1}$, and add edge uu' to H.

  We claim that H is a spanning tree of G. First, observe that H is spanning by construction. Next, we show that H is connected and acyclic. We prove this by induction on i. Specifically, we prove by induction on i that the subgraph of H

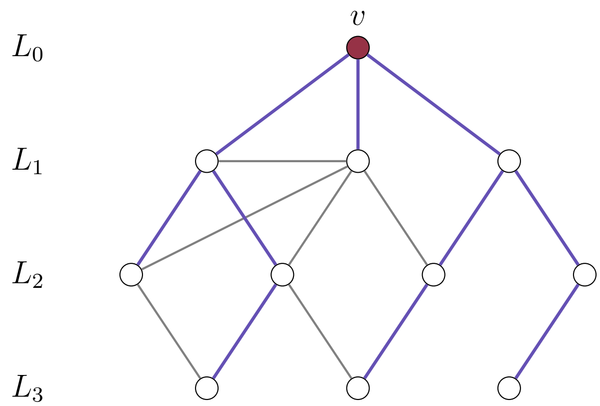


FIGURE 9. A figure illustrating the construction of a spanning tree. This is called a **breadth-first search** spanning tree.

induced by $\bigcup_{j=0}^{i} L_j$ contains a path from u to v for every vertex v in L_i and has no cycles.

The base case, $L_0 = \{v\}$, is true because the single vertex graph contains a (trivial) path from v to v and is acyclic.

Next, assume that the statement is true for $i \geq 1$, and consider the statement for $i+1$. Let u be a vertex of L_{i+1}. Then, there is a neighbor u' of u in L_i. By the induction hypothesis, there is a path P from u' to v in the subgraph of H induced by $\bigcup_{j=0}^{i} L_j$. Now, adding edge $u'u$ to P is a path from u to v in the subgraph of H induced by $\bigcup_{j=0}^{i+1} L_j$.

Finally, suppose for a contradiction that the subgraph induced by $\bigcup_{j=0}^{i+1} L_j$ contains a cycle C. By the induction hypothesis, there is no cycle in the subgraph induced by $\bigcup_{j=0}^{i} L_j$. Therefore, C must contain a vertex u from L_{i+1}. But by the construction of H, every vertex of L_{i+1} has degree one in the subgraph induced by $\bigcup_{j=0}^{i+1} L_j$, a contradiction.

This completes the proof. □

We can also do a simpler proof using the characterization of trees.

Simple proof of Claim 2.2. Let G be a connected graph. Since G is connected, G contains a connected spanning subgraph (itself). Therefore, we can choose an edge-minimal connected spanning subgraph H of G.

Suppose for a contradiction that there is an edge e of H such that $H - e$ is connected. Then, $H - e$ is a connected spanning subgraph of G, contradicting that H is edge-minimal with these properties. Therefore, H is connected but for every edge e of H the graph $H - e$ is disconnected. Now, by Theorem 2.1 (c) $\rightarrow$ (a), H is a tree. Since H was chosen to be spanning, it follows that H is a spanning tree. □

- We also proved question 2 above:

Claim 2.3. A graph G is bipartite if and only if G contains no odd cycle.

Proof. First, observe that a graph is bipartite if and only if every connected component of the graph is bipartite. Therefore, we may assume that the graph G is connected.

Assume that G is bipartite with bipartition $\{A, B\}$. We will show that G contains no odd cycles. Suppose for a contradiction that G contains a cycle C with $2k + 1$ vertices $v_1, v_2, \ldots, v_{2k+1}$ and edges $\{v_1v_2, v_2v_3, \ldots, v_{2k}v_{2k+1}, v_{2k+1}v_1\}$. Up to symmetry between A and B, assume that $v_1 \in A$. Then, since v_1v_2 is an edge, it follows that $v_2 \in B$. Similarly, since the vertices of the cycle must alternate between A and B, it follows that every vertex v_i with i even is in B and every vertex v_i with i odd is in A. But now v_1 and v_{2k+1} are both in A, contradicting that $\{A, B\}$ is a bipartition of G.

For the other direction, assume that G contains no odd cycles. Let T be a spanning tree of G and pick a vertex v of T. Define A to be the set of vertices whose distance from v in T is odd, and define B to be the set of vertices whose distance from v in T is even.

We claim that this is a valid bipartition of G, i.e. every edge of G has one end in A and one end in B. Suppose for a contradiction that G contains an edge uw with u, w both in the same part. Assume up to symmetry between A and B that $u, w \in A$. Let P be the unique path from u to w in T (this unique path exists because of Theorem 2.1 (b)). The vertices in P alternate between vertices in A and vertices in B and both ends are in A, so P has even length. Now, the path P plus the edge uw forms an odd cycle in G, a contradiction. This completes the proof. $\square$

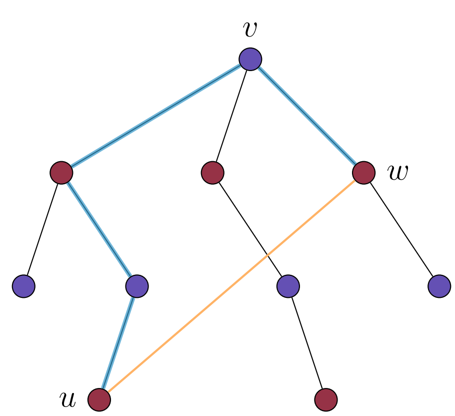


FIGURE 10. An illustration of the proof of the second direction of Claim 2.3. The path P is shown in bright blue. The path P has even length since its vertices alternate colors and its ends have the same color.

2.2. **January 15th.**

- Minor
- Subdivision, contraction
- Topological minor
- Independent set, clique

Let G and X be graphs. We say that **G is an IX** if there is a partition $\{V_x \mid x \in V(X)\}$ of $V(G)$ satisfing the following:

- V_x is a connected subset of G for every $x \in V(X)$;
- for every pair $x, y \in V(X)$, the edge xy is in X if and only if G contains an edge with one end in V_x and one end in V_y.

The sets V_x are called the **branch sets** of the IX. If G is an IX, then we say that X is a **contraction minor** of G, and that X is formed from G by contracting each branch set V_x.

If a graph G contains an IX as a subgraph, then X is a **minor** of G. The IX is the **model** of X in G.

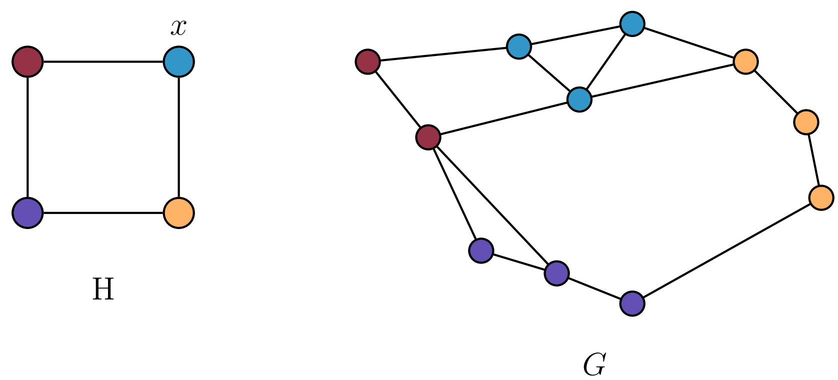


FIGURE 11. In this figure, the graph G on the right is an IH, where H is the graph on the left. The vertex-partition of G giving the branch sets is indicated by the vertex colors. For example, V_x for the labeled vertex x on the left consists of the set of the blue vertices of G.

Let X be a graph. A **subdivision** of X is a graph G formed from X by replacing edges of X with paths of length at least one. If G is a subdivision of X, then we say that **G is a TX**. If a graph G contains a TX as a subgraph, then X is a **topological minor** of G.

Let G be a graph. A vertex-subset I of G is **independent** if there is no edge of G with both endpoints in I. A **maximum independent set** of G is an independent set of G of maximum size. We denote the maximum independent set of a graph G by $\alpha(G)$.

A vertex-subset K of G is a **clique** if for every pair of vertices x, y in K, the edge xy is in G. The clique consisting of n vertices is denoted K_n.

2.2.1. **Class plan.** In class we will go over the following problems.

1. A **cycle decomposition** of a graph G is a set of subgraphs $\{C_1, \ldots, C_k\}$ of G such that each subgraph C_i is a cycle and each edge of G appears in exactly one subgraph C_i. Prove that a graph G admits a cycle decomposition if and only if the degree of every vertex of G is even.
2. A **Eulerian tour** of a graph G is a closed walk of G that uses each edge of G exactly once. A graph is **Eulerian** if it has an Eulerian tour. Prove that a connected graph G is Eulerian if and only if G the degree of every vertex of G is even.
3. What is the relationship between subgraphs, induced subgraphs, minors, subdivisions, and topological minors?
4. Prove that a graph G contains a graph X as a minor if and only if X can be formed from G by deleting vertices, deleting edges, and contracting edges.
5. Describe in a word the class of graphs consisting of all graphs that do not have a triangle minor.
6. Prove that if a graph G contains a graph X as a minor and $\Delta(X) \leq 3$, then G contains X as a topological minor.

2.2.2. **References.** Optional references for today's class:

- Diestel Section 1.8
- Diestel Graph Theory Lecture 5 from minute 30:00 to minute 1:17:00.

2.2.3. **Class log.**

- We proved question 1 above:

  Claim 2.4. A graph G admits a cycle decomposition if and only if the degree of every vertex of G is even.

  Proof. First, suppose that G admits a cycle decomposition $\{C_1, \ldots, C_k\}$. Let v be a vertex of G and let $\mathcal{C}_v \subseteq \{C_1, \ldots, C_k\}$ be the set of cycles containing v. Since every edge of G is in exactly one subgraph C_i, the degree of v in G is exactly $2 \cdot |\mathcal{C}_v|$, which is always even.

Now, assume that the degree of every vertex of G is even. We prove by induction on the number of edges of G that G admits a cycle decomposition. The base case is the graph with no edges, which trivially satisfies the statement.

Next, let G be a graph with a positive number of edges where every vertex has even degree, and assume that the statement holds for every graph G' with fewer edges than G, i.e. for all graphs G' with $|E(G')| < |E(G)|$. Let H be the subgraph of G induced by the vertices of degree at least two. Since every graph with minimum degree at least two contains a cycle, H contains a cycle C. Let $G' = G - E(C)$ be the graph formed from G by deleting the edges of C. The degree of a vertex v is the same in G' as in G' if $v \notin V(C)$, and the degree of v in G' is exactly two less than the degree of v in G if $v \in V(C)$. Therefore, every vertex of G' has even degree. By the induction hypothesis, G' admits a cycle decomposition $\{C_1, \ldots, C_k\}$. It follows that $\{C, C_1, \ldots, C_k\}$ is a cycle decomposition of G. This completes the proof. $\square$

- We also proved question 2 above. Note that I changed the question slightly to assume that we only care about connected graphs.

Claim 2.5. A connected graph G is Eulerian if and only if G the degree of every vertex of G is even.

Proof. First, assume that G is Eulerian and let $W = v_1e_1v_2e_2v_3e_3 \ldots v_ke_kv_1$ be an Eulerian tour of G. Let v be a vertex of G. Then, v is incident with two edges for each time that v appears in W. Since every edge appears exactly once in W, it follows that v has even degree.

For the other direction, we prove by induction on the number of edges of G that every connected graph G where every vertex has even degree is Eulerian. The base case is the graph with one vertex and no edges, which is trivially Eulerian. Next, let G be a connected graph with a positive number of edges where every vertex has even degree, and assume that the statement holds for every graph G' with fewer edges than G. Since G has minimum degree two, G contains a closed walk where every each is used at most once (e.g., a cycle). Let W be the longest closed walk in G that uses each edge at most once.

Suppose for a contradiction that there is an edge of G not in W. Since G is connected, we can choose an edge e not in W such that one end u of e is in W. Let C be the connected component of $G - E(W)$ containing e. Since every vertex has even degree in W and every vertex has even degree in G, it follows that every vertex has even degree in $G - E(W)$. In particular, C is a connected graph with $|E(C)| < |E(G)|$ and such that every vertex has even degree. By the inductive hypothesis, C contains an Eulerian tour T.

Now, W and T are edge-disjoint closed walks that both contain u and that both use each edge of G at most once. Therefore, we can concatenate them into a closed walk that uses each edge of G at most once and that is longer than W, a contradiction. Therefore, W is an Eulerian tour of G. $\square$

3. Matchings

3.1. January 20th.

- Matching, perfect matching
- k-factor
- Alternating path, augmenting path
- Vertex cover

Let G be a graph. A **matching** in G is a set $M \subseteq E(G)$ of edges of G such that no two edges in M share an end. Given a graph G and a matching M of G, a vertex v of G is **matched** if v is incident to an edge of M, and **unmatched** otherwise. A matching M of a graph G is **perfect** if every vertex of G is matched, i.e. if every vertex of G is incident with an edge of M.

A matching M of a graph G is a **maximum matching** if M has the maximum number of edges of any matching of G. A matching M of a graph G is a **maximal matching** if there is no matching M' of G such that $M \subsetneq M'$.

For an integer $k \geq 1$, a graph is called **k-regular** if every vertex has degree k. For an integer $k \geq 1$ and a graph G, a **k-factor of G** is a k-regular spanning subgraph H of G, i.e. a subgraph H of G with $V(H) = V(G)$ and where every vertex has degree k in H. A 1-factor is equivalent to a perfect matching. (Why?)

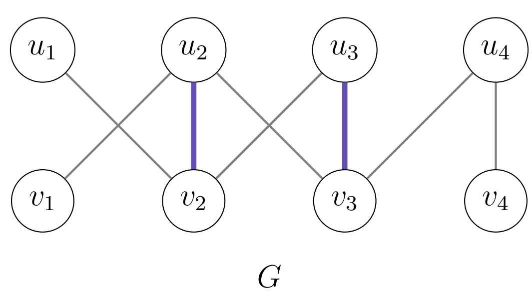


Figure 12. An example of a graph G and a matching M shown in purple. An example of an alternating path is $P_1 = u_1$-v_2-u_2-v_3-u_3. An example of an augmenting path is $P_2 = v_1$-u_2-v_2-u_3-v_3-u_4. The matched vertices are $\{u_2, v_2, u_e, v_3\}$ and the unmatched vertices are $\{u_1, v_1, u_4, v_4\}$.

Let G be a graph and M a matching of G. A path P of G is a **alternating path** if the first vertex of P is unmatched, and the edges of P alternate between edges in M and edges not in M. If the last vertex of an alternating path is also unmatched, then it is called an **augmenting path**.

Let G be a graph. A vertex-set $S \subseteq V(G)$ of G is a **vertex cover** of G if every edge of G is incident with a vertex of S. A **minimum vertex cover** of G is a vertex cover of G with the minimum number of vertices of any vertex cover of G.

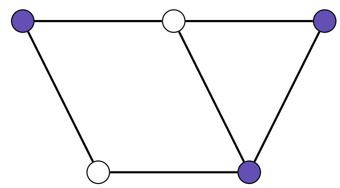


FIGURE 13. An example of a vertex cover shown in purple.

3.1.1. **Class plan.** In class we will go over the following problems.

1. Let G be a graph and M a matching of G. Prove that if G contains an augmenting path, then M is not a maximum matching.

2. Prove König's Theorem:

   Theorem 3.1 (König's Theorem)**.** Let G be a bipartite graph. The size of a maximum matching of G is equal to the size of a minimum vertex cover of G.

3. Prove Hall's Theorem:

   Theorem 3.2 (Hall's Theorem)**.** Let G be a bipartite graph with bipartition $\{A, B\}$ such that $|A| \leq |B|$. Then, G contains a matching that matches every vertex of A if and only if $|N(S)| \geq |S|$ for every set $S \subseteq A$.

3.1.2. **References.** Optional references for today's class:

- Diestel Section 2.1
- Diestel Graph Theory Lecture 6

3.1.3. **Class log.**

- We went over the proofs of the three problems listed above.
- König's Theorem is Theorem 2.1.1 in Diestel's book; the proof there on page 39 is the same proof we did in class.
- See the next day's class log for notes on Hall's Theorem.

3.2. **January 22nd.**

- Stable matching
- Odd component

Let G be a bipartite graph with bipartition $\{A, B\}$. Suppose that every vertex has *preferences* among its neighbors. Formally, each vertex v has an ordering $\leq_v$ which orders the edges incident with v. This ordering represents the preferences of v.

A matching M in G is a **stable matching** if for every edge e in $E \setminus M$, i.e. for every edge e not included in the matching M, there is an edge f in M such that e and f share an endpoint v and v prefers f to e, i.e. $e <_v f$. Intuitively, a matching M is *unstable* if there is some edge $e = uv$ not in M such that both u and v either are not matched by M or prefer edge e to the edge that they are incident to in M.

An **odd component** of a graph G is a connected component of G with an odd number of vertices. Given a graph G, let $q(G)$ denote the number of odd components of G.

3.2.1. **Class plan.** In class we will go over the following problems.

1. Continue the proofs of Hall's Theorem.

2. Discuss Tutte's Theorem:

   Theorem 3.3 (Tutte's Theorem)**.** A graph G has a perfect matching if and only if $q(G - S) \leq |S|$ for every vertex-subset $S \subseteq V(G)$.

3. A graph G is **cubic** if every vertex of G has degree 3. Use Tutte's Theorem to prove that every cubic graph with no cut edge has a perfect matching.

4. Prove the Gale-Shapley Theorem:

   Theorem 3.4 (Gale-Shapley Theorem)**.** Let G be a bipartite graph equipped with vertex preferences. Then, G has a stable matching.

3.2.2. **References.** Optional references for today's class:

- Diestel Section 2.2
- Diestel Graph Theory Lecture 7 up to minute 53:00.

3.2.3. **Class log.**

- We went over items 1, 2, and 3 above, but not the proof of the Gale-Shapley Theorem.
- We went over the following proof:

Claim 3.5. Let G be a cubic graph with no cut edge. Then G has a perfect matching.

Proof. We show that G satisfies Tutte's condition. Then, by Tutte's theorem, G has a perfect matching.

Let $S \subseteq V(G)$ be a vertex-subset of G, and let C be an odd component of $G - S$. Since C has an odd number of vertices and every vertex has degree 3, the total sum of degrees in G of vertices in C is odd. The edges that contribute to this degree sum are the edges with both ends in C and the edges with one end in C and one end in S. The edges with both ends in C contribute an even number to the degree sum, which means that an odd number of edges have one end in C and one end in S. Since G has no cut edge, there are at least three edges with one end in C and one end in S.

We have shown that the number of edges with one end in S is at least $3q(G-S)$. Since G is cubic, we also know that the number of edges with one end in S is at most $3|S|$. Therefore, $3q(G - S) \leq 3|S|$. This shows that G satisfies Tutte's condition, as desired. $\square$

- We also went over the induction proof of Hall's Theorem.

Claim 3.6. Let G be a bipartite graph with bipartition (A, B) satisfying $|A| \leq |B|$. Assume that G satisfies Hall's condition. Then, G has a matching with every vertex in A matched.

Proof. We proceed by induction on $|A|$. The base case is when $|A| = 1$. Let $A = \{a\}$. Then, by Hall's condition, we can choose an edge ab with $b \in B$, and this edge is a matching with every vertex in A matched.

Next, let $|A| = n > 1$, and assume that the statement holds for all graphs with $|A| < n$. We consider two cases.

In the first case, suppose that $|N(S)| \geq |S|+1$ for every set $S \subseteq A$ (so something even stronger than Hall's condition holds). Let a be a vertex of A. By Hall's condition, there is an edge ab with $b \in B$. Let G' be the graph formed from G by deleting the vertices a and b, with bipartition (A', B') where $A' = A - \{a\}$ and $B' = B - \{b\}$. Observe that G' satisfies Hall's condition: for every set $S' \subseteq A'$, the neighborhood of S' in G' is the same as the neighborhood of S' in G, except possibly with one vertex, b, missing. But since G satisfies that $|N(S)| \geq |S| + 1$ for all sets $S \subseteq A$, it holds that $|N(S')| \geq |S'|$ in G'.

Now, by the induction hypothesis, G' has a matching M' with every vertex of A' matched. Adding edge ab to M' gives a matching M of G with every vertex of A matched. This completes the first case.

In the second case, there exists a set $S \subseteq A$ such that $|N(S)| = |S|$. Let G' be the subgraph of G induced by $S \cup N(S))$, with $A' = S$ and $B' = N(S)$. Let $G'' = G - G'$, with $A'' = A - A'$ and $B'' = B - B'$.

First, observe that since G satisfies Hall's condition, G' also satisfies Hall's condition. Therefore, by the induction hypothesis, there is a matching M' such that every vertex of A' is matched.

Next, observe that G'' also satisfies Hall's condition. Let $S'' \subseteq A''$ and consider $N_{G''}(S'')$ (where this notation means the neighborhood in G''). Since G satisfies Hall's condition, it follows that $|N(S \cup S'')| \geq |S \cup S''|$ in G. Now, $|N(S \cup S'')| = |N(S)| + |N_{G''}(S'')|$, and since S and S'' are disjoint, $|S \cup S''| = |S| + |S''|$. Recall that S was chosen so that $|N(S)| = |S|$. Therefore, $|N_{G''}(S'')| \geq |S''|$, so G'' satisfies Hall's condition. By the induction hypothesis, there is a matching M'' such that every vertex of A'' is matched.

Finally, let $M \coloneqq M' \cup M''$ be the matching formed by the union of M' and M''. Now, M is a matching of G with every vertex of A matched. This completes the proof. $\square$

4. Connectivity

4.1. January 27th.

- Block
- Separating set
- (Review) cut vertex, cut edge, k-connected,
- (Review) K_4, subdivision, topological minor

Let G be a graph. A **block** of G is a maximal connected subgraph H of G such that H has no cut vertex. A subgraph H of G is a block of G if and only if H is one of the following:

- a maximal 2-connected subgraph of G,
- a cut edge of G, or
- an isolated vertex of G.

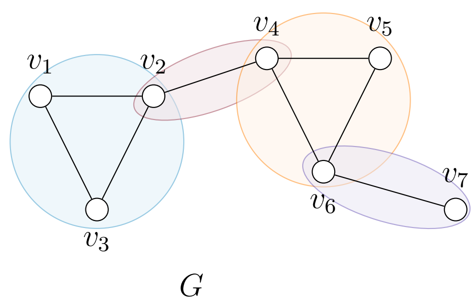


Figure 14. The graph G with four blocks.

Let G be a graph and let $A \subseteq V(G)$ and $B \subseteq V(G)$ be vertex-subsets of G. A vertex-subset $X \subseteq V(G)$ **separates A and B** if every path of G with one end in A and one end in B contains a vertex of X.

4.1.1. Class plan.

In class we will go over the following problems.

1. Define the **block graph** of a graph.

2. Prove that for every connected graph G, the block graph of G is a tree.

3. Prove that every 3-connected graph contains K_4 as a topological minor.

4.1.2. References.

Optional references for today's class:

- Diestel Section 3.1, parts of 3.2
- Diestel Graph Theory Lecture 8 up to around minute 40:00.

4.1.3. **Class log.**

- We went over the items listed above.
- We also proved the following.

 Claim 4.1. A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed. (An H-path is a path P whose ends are in H and whose internal vertices are not in H.)

 Proof. Every graph constructed as described is 2-connected. Conversely, suppose that G is 2-connected. Let H be a maximal subgraph of G created from a cycle as described in the statement. We claim that $H = G$.

 First, suppose that e is an edge of G with both ends in H. Then, e is an H-path, so by the maximality of H, it follows that e is in H. Therefore, H is an induced subgraph of G.

 Suppose for a contradiction that $V(H) \neq V(G)$. Since G is connected, there exists a vertex v such that v is not in $V(H)$ but v has a neighbor u in H. Since G is 2-connected, the graph $G - u$ formed from G by deleting u is still connected. Therefore, there is a path from v to $H - u$ in $G - u$. Let P be the shortest such path; this implies that the end of P in H is the only vertex of P in H. Now, the path Q formed by adding edge uv to P is an H-path, contradicting the maximality of H.

 We have proven that H is an induced subgraph of G and that $V(H) = V(G)$. Therefore, $H = G$. This completes the proof. □

- I mentioned in class that Claim 4.1 can be used to prove the last problem on the homework, but please do note that there is also a direct proof of the last problem that does not depend on Claim 4.1 and only uses the definition of 2-connected. You may prefer to look for this solution, to test your understanding of the concepts.
- I also wrote down the augmenting path proof of the hard direction of Hall's Theorem, since we did not get the chance to go over it again in class.

 Claim 4.2. Let G be a bipartite graph with bipartition (A, B) satisfying $|A| \leq |B|$. Assume that G satisfies Hall's Condition. Then, G has a matching M such that every vertex of A is matched by M.

 Proof. Let M be a maximum matching of G. Suppose for a contradiction that some vertex a in A is not matched by M. Let $\mathcal{P}$ be the set of all alternating paths of G starting from M. Let $A' \subseteq A$ and $B' \subseteq B$ be the vertices of A and B, respectively, that are the endpoints of paths in $\mathcal{P}$.

 We claim that every vertex of B' is matched by M. Suppose for a contradiction that some vertex b of B' is not matched by M. Since b is in B', it follows that b is the end of an alternating path P starting from a. But now P is an augmenting path since b is unmatched by M, which contradicts that M is a maximum matching. Therefore, every vertex of B' is matched by M.

For every edge $a'b$ of M with b in B', the alternating path P that starts at a and ends at b can be extended by edge $a'b$ to an alternating path starting at a and ending at a'. Therefore, a' is in A', and so $|A' - a| \geq |B'|$. Since G satisfies Hall's condition, $|N(A')| \geq |A'|$, so $N(A')$ contains some vertex not in B'. Therefore, there is an edge xy of G with $x \in A'$ and $y \in B \setminus B'$.

Let P_x be the alternating path starting at a and ending at x (observe that P_x exists because x is in A'). Because P_x is an alternating path, the last edge of P_x is in M. Also, since y is not in B', we know that y cannot be reached by an alternating path from a, so y is not contained in P_x. Now, we can extend path P_x by adding edge xy. This forms an alternating path starting from a and ending at y, contradicting that y is not in B'. This completes the proof. $\square$

4.2. **January 29th.**

- Disjoint paths
- Edge-connected, edge-connectivity

Let G be a graph. Suppose that $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ is a collection of paths of G. The paths in $\mathcal{P}$ are **vertex-disjoint** if every vertex of G appears in at most one path of $\mathcal{P}$.

The paths in $\mathcal{P}$ are **edge-disjoint** if every edge of G appears in at most one path of $\mathcal{P}$. Observe that if a set of paths are vertex-disjoint, then they are also edge-disjoint (but the converse is false).

Let $\ell \geq 0$ be an integer. If G is a graph with more than one vertex and $G - F$ is connected for every edge-subset $F \subseteq E(G)$ with $|F| \leq \ell$, then we say that G is **ℓ-edge-connected**.

The maximum integer ℓ such that G is ℓ-edge-connected is called the **edge-connectivity** of G and is denoted $\lambda(G)$.

4.2.1. **Class plan.** In class we will go over the following problems.

1. Prove Menger's Theorem:

   Theorem 4.3 (Menger's Theorem)**.** Let G be a graph and let $A, B \subseteq V(G)$ be vertex-subsets of G. Then the minimum number of vertices separating A from B in G is equal to the maximum number of vertex-disjoint A–B paths in G.

4.2.2. **References.** Optional references for today's class:

- Diestel Section 3.3
- Diestel Graph Theory Lecture 9 up to 1:14:00.

4.3. **Midterm important terms.**

- Graph, vertex, edge
- Adjacency, incidence, degree, neighborhood
- Minimum degree, maximum degree, average degree
- Subgraph, induced subgraph, spanning subgraph
- Graph isomorphism
- Complement of a graph
- Walk, trail, path
- Connected, disconnected
- Cycle
- Acyclic, forest, tree, leaf
- Cut vertex, cut edge
- k-connected
- Spanning tree
- Bipartite graph
- Independent set, vertex cover
- **Matching, perfect matching**

5. Planar graphs

5.1. **February 4th.**

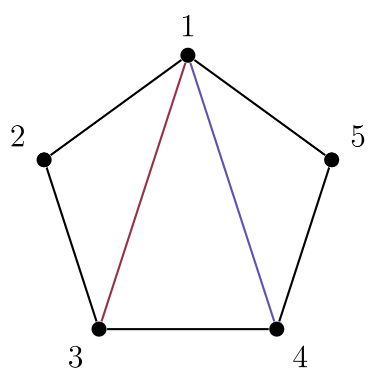
- Planar graph
- Planar representation
- Face, face boundary
- Length of a face boundary

A graph G is **planar** if G can be drawn in the plane such that no edge crosses another edge or a vertex. Such a drawing of a graph G in the plane is called a **planar representation** of G.

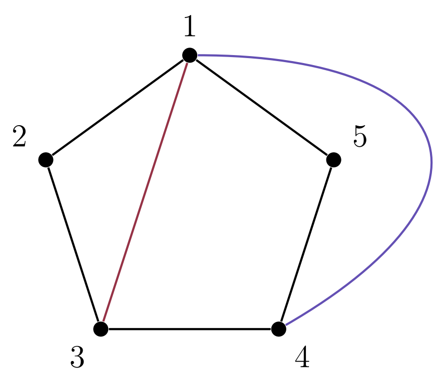
A **face** of a planar representation of G is a (connected) region of the plane formed by deleting (the drawing of) G. The **boundary** of a face is the subgraph of G which touches the face in the planar representation.

Every face boundary of a connected planar representation can be written as a closed walk. If G is disconnected, then every face boundary of a planar representation of G except the boundary of the infinite face can be written as a closed walk; the infinite face boundary can be written as the sum of k closed walks, where k is the number of connected components of G.

The **length** of a face boundary is the length of the shortest closed walk representing the face boundary.



(a) Faces of lengths 3, 3, 3, 5



(b) Faces of lengths 3, 3, 4, 4

Figure 15. The same graph $G = C_5$ drawn with two different planar representations. The faces and face boundaries depend on the planar representation, though the number of faces and the sum of the lengths of the boundaries do not (see Questions 4 and 5 below).

5.1.1. **Class plan.** In class we will go over the following problems.

1. Observations:
   - Consider a planar representation of a graph G. Let e be an edge of G. If e is in a cycle of G, then e is in exactly two face boundaries. If e is not in a cycle of G, then e is in exactly one face boundary.
   - If H is a subgraph of G, then a planar representation of G induces a planar representation of H.
   - A planar forest has exactly one face.
2. If G is a planar graph such that two distinct faces of G have the same boundary, then G is a cycle.
3. If G is planar and 2-connected, then every face boundary of G is a cycle.
4. Let G be a connected planar graph and let n, e, and f denote the number of vertices, edges, and faces of G, respectively. Then,
$$n - e + f = 2 \quad \text{(Euler's formula)}.$$
5. Let G be a planar graph and, for a face f of G, let $\ell(f)$ denote the length of the face boundary of f. Then,
$$\sum_{f \text{ face of } G} \ell(f) = 2|E(G)|.$$

5.1.2. **References.** Optional references for today's class:

- Diestel Section 4.2
- Diestel Graph Theory Lecture 10 and Lecture 11 from 45:00

5.2. **February 10th.**

- Planar triangulation
- Maximally planar graph
- Maximal planar representation
- Minor-closed graph class
- Outerplanar graph

A representation of a planar graph G is a **planar triangulation** if every face of G (including the outer face) is bounded by a triangle.

A graph G is a **maximally planar graph** if G is planar but adding edge xy to G is not planar for any pair of non-adjacent vertices $x, y \in V(G)$. A planar representation of G is a **maximal planar representation** if edge xy cannot be added to the planar representation without creating a crossing for every pair of non-adjacent vertices $x, y \in V(G)$.

A graph class $\mathcal{C}$ is **minor-closed** if for every graph G in $\mathcal{C}$, the class $\mathcal{C}$ contains every minor of G.

A graph is **outerplanar** if it can be drawn in the plane without crossings such that every vertex is on the boundary of the outer face.

5.2.1. **Class plan.** In class we will go over the following problems.

1. Prove that if G is a planar graph with at least three vertices, then $|E(G)| \leq 3|V(G)| - 6$.

2. Prove that if G is a triangle-free planar graph, then $|E(G)| \leq 2|V(G)| - 4$.

3. Prove that if G is a planar graph with at least three vertices, then a planar representation of G is maximal if and only if G is a planar triangulation.

4. Prove that a planar graph G is maximally planar if and only if $|E(G)| = 3|V(G)| - 6$.

5. Prove that K_5 and $K_{3,3}$ are not planar.

6. Observe that planar graphs are a minor-closed family.

7. State the following theorem:

   Theorem 5.1 (Kuratowski-Wagner Theorem)**.** A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

8. (Time permitting) Prove that in a 3-connected graph, the face boundaries are precisely the non-separating cycles.

5.2.2. **References.** Optional references for today's class:

- Diestel Section 4.2 and parts of 4.4
- Diestel Graph Theory Lecture 11 and parts of Lecture 12 (though it's a bit more advanced)

5.3. **Class log.**

- We went over problems 1-4 above. We will cover 5-7 at the start of next class.
- Here is a proof of problem 3 above.

Claim 5.2. A planar representation of a graph G with at least three vertices is maximal if and only if it is a planar triangulation.

Proof. Fix a planar representation of G. First, suppose that the planar representation is maximal. Suppose for a contradiction that some face boundary F of G has two vertices u, v which are non-adjacent in G. Then, edge uv can be added to the planar representation by drawing edge uv through face F, contradicting the maximality of the planar representation. Therefore, every face boundary is complete. The only complete graphs which can form face boundaries are K_2 and K_3, so every face boundary is K_2 or K_3. Observe that if a graph has a maximal planar representation then it is connected; so, since G has at least three vertices, K_2 is not a face boundary of G. It follows that every face boundary of G is a triangle.

Conversely, suppose that the planar representation is a planar triangulation. Let u and v be non-adjacent vertices of G, and suppose for a contradiction that edge uv can be added to the representation without creating a crossing. Then, edge uv is contained within some face F, and u and v are on the boundary of F. But every face boundary is a triangle, so no face boundary contains two non-adjacent vertices, a contradiction.

This completes the proof. $\square$

6. Graph coloring

6.1. February 12th.

- Vertex coloring
- Chromatic number, k-coloring, k-colorable
- Color class

A **vertex coloring** of a graph G, often also called a **proper coloring**, is a map $c : V(G) \to S$ from the vertices of G to a set of **colors** S such that $c(v) \neq c(w)$ if v and w are adjacent, i.e. the ends of every edge have two different colors. We typically take the set of colors to be the set of integers $\{1, \ldots, k\}$, where k is the total number of colors (or elements of S).

Let G be a graph. If there is a vertex coloring $c : V(G) \to \{1, \ldots, k\}$, then c is called a **k-coloring**, and G is called **k-colorable**. The **chromatic number** of a graph G is the minimum k for which G is k-colorable. The chromatic number of a graph G is denoted $\chi(G)$.

The **color classes** of a coloring $c : V(G) \to \{1, \ldots, k\}$ of G are the vertex-subsets of G formed by partitioning the vertices of G by their assigned color. For example, $\{v \in V(G) \mid c(v) = 1\}$ is the color class corresponding to the color 1. A k-coloring has k color classes. Observe that by the definition of a vertex coloring, each color class of a coloring is an independent set.

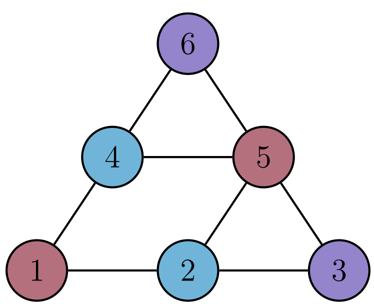


FIGURE 16. An example of a graph and a 3-coloring, with the colors red, blue, and purple. The three color classes are $\{2, 4\}$, $\{1, 5\}$, and $\{3, 6\}$.

6.1.1. Class plan.

In class we will go over the following problems.

1. Which graphs are 2-colorable?

2. Give a bound on $\chi(G)$ in terms of $\alpha(G)$ (the **independence number** of G) and $\omega(G)$ (the **clique number** of G).

3. Give a bound on $\chi(G)$ in terms of $\Delta(G)$, the maximum degree of G.

4. Go over the definition of *degeneracy* and its relationship to chromatic number.

6.1.2. **References.** Optional references for today's class:

- Diestel Section 5.2
- Diestel Graph Theory Lecture 13

6.2. **Class log.**

Claim 6.1. For every graph G, we have $\chi(G) \geq \omega(G)$ and $\chi(G)\alpha(G) \geq |V(G)|$.

Proof. Fix a graph G. Suppose that K_t is a subgraph of G. Then, every pair of vertices u, v in the K_t subgraph are adjacent in G, so no two vertices in K_t can be colored the same color in a proper coloring. This shows that $\chi(G) \geq \omega(G)$.

Next, suppose we have a proper coloring of G with $\chi(G)$ colors. The color classes form a partition of $V(G)$ and each color class is an independent set. Therefore, $V(G)$ can be partitioned into $\chi(G)$ independent sets, each of size at most $\alpha(G)$. It follows that $\chi(G)\alpha(G) \geq |V(G)|$. $\square$

6.3. **February 19th.**

- Degeneracy
- Edge coloring, k-edge-coloring
- Chromatic index

For a positive integer k, a graph G is **k-degenerate** if every subgraph of G has a vertex of degree at most k. The **degeneracy** of a graph G is the minimum k for which G is k-degenerate.

An **edge coloring** of a graph G is a map $c : E(G) \to S$ from the edges of G to a set of **colors** S such that $c(e) \neq c(f)$ for any pair of edges e, f of G that share an endpoint. If there is an edge coloring $c : E(G) \to \{1, \dots, k\}$ of a graph G, then c is called a **k-edge-coloring**, and G is called **k-edge-colorable**.

The smallest k for which a graph G is k-edge-colorable is called the **edge-chromatic number** or the **chromatic index** of G. We denote the chromatic index of G by $\chi'(G)$.

6.3.1. **Class plan.** In class we will go over the following problems.

1. Prove that planar graphs have chromatic number at most 6.
2. Go over the following theorem.

   Theorem 6.2 (Brooks' Theorem)**.** Let G be a connected graph. If G is neither complete nor an odd cycle, then
   $$\chi(G) \leq \Delta(G).$$
3. Prove that every k-chromatic graph has a k-chromatic subgraph of minimum degree at least $k - 1$.
4. Prove that $\chi'(G) = \Delta(G)$ for every bipartite graph G.
5. Prove that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every graph G.

6.3.2. **References.** Optional references for today's class:

- Diestel Sections 5.2 and 5.3
- Diestel Graph Theory Lecture 14 from 40:00.

6.4. **Midterm 2 important terms.** Everything through the end of class on Thursday, February 19th, is in scope for Midterm 2. Here are the most important terms.

- Tutte's theorem
- Connectivity, edge-connectivity
- Menger's Theorem (statement)
- Planar graphs
- Faces, face boundaries of planar graphs
- Euler's formula and the edge bound for planar graphs
- Maximal planar graphs
- Planar triangulations
- Vertex coloring, chromatic number
- Edge coloring, chromatic index
- Minors, topological minors
- Minor-closed graph class